

NONEXPANSIVE \mathbb{Z}^2 -SUBDYNAMICS AND NIVAT'S CONJECTURE

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ABSTRACT. For a finite alphabet \mathcal{A} and $\eta: \mathbb{Z} \rightarrow \mathcal{A}$, the Morse-Hedlund Theorem states that η is periodic if and only if there exists $n \in \mathbb{N}$ such that the block complexity function $P_\eta(n)$ satisfies $P_\eta(n) \leq n$, and this statement is naturally studied by analyzing the dynamics of a \mathbb{Z} -action associated to η . In dimension two, we analyze the subdynamics of a \mathbb{Z}^2 -action associated to $\eta: \mathbb{Z}^2 \rightarrow \mathcal{A}$ and show that if there exist $n, k \in \mathbb{N}$ such that the $n \times k$ rectangular complexity $P_\eta(n, k)$ satisfies $P_\eta(n, k) \leq nk$, then the periodicity of η is equivalent to a statement about the expansive subspaces of this action. As a corollary, we show that if there exist $n, k \in \mathbb{N}$ such that $P_\eta(n, k) \leq \frac{nk}{2}$, then η is periodic. This proves a weak form of a conjecture of Nivat in the combinatorics of words.

1. INTRODUCTION

1.1. Periodicity and complexity. Given a finite alphabet \mathcal{A} , if $\eta \in \mathcal{A}^{\mathbb{Z}}$ is an infinite word, the *block complexity function* $P_\eta(n)$ is defined to be the number of distinct words of length n appearing in η . The word $\eta = (\eta_n)_{n \in \mathbb{Z}}$ is said to be *periodic* if there exists an integer $m \in \mathbb{N}$ such that $\eta_n = \eta_{n+m}$ for all $n \in \mathbb{Z}$. The classical Morse-Hedlund Theorem gives the relationship between these two notions:

Theorem 1.1 (Morse-Hedlund [12]). *The infinite word $\eta \in \mathcal{A}^{\mathbb{Z}}$ is periodic if and only if there exists an integer $n \geq 1$ such that $P_\eta(n) \leq n$.*

For $\eta \in \mathcal{A}^{\mathbb{Z}^d}$, the $(n_1 \times \dots \times n_d)$ -*block complexity function* $P_\eta(n_1, \dots, n_d)$ is the number of distinct $n_1 \times \dots \times n_d$ blocks occurring in η . Periodicity also has a natural higher dimensional generalization, and we say that the infinite word $\eta = (\eta_{\vec{n}})_{\vec{n} \in \mathbb{Z}^d}$ is *periodic* if there exists a *period vector*, meaning a vector $\vec{m} \in \mathbb{Z}^d$ such that $\eta_{\vec{n}} = \eta_{\vec{n}+\vec{m}}$ for all $\vec{n} \in \mathbb{Z}^d$.

Nivat conjectured that there is a simple analog of the Morse-Hedlund Theorem in two dimensions:

Conjecture (Nivat [13]). *For $\eta \in \mathcal{A}^{\mathbb{Z}^2}$, if there exist integers $n, k \geq 1$ such that $P_\eta(n, k) \leq nk$, then η is periodic.*

In a first step toward the conjecture, Sander and Tijdeman [17] showed that if there is some n such that $P_\eta(n, 2) \leq 2n$ (or such that $P_\eta(2, n) \leq 2n$), then η is periodic. Soon after, Epifanio, Koskas and Mignosi [8] proved a weak version of the conjecture showing that if $P_\eta(n, k) \leq \frac{nk}{144}$ for some n and k , then η is periodic; Quas and Zamboni [14] improved the constant to $\frac{1}{16}$.

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Conversely, Sander and Tijdeman [15, Example 5] found counterexamples to the analog of Nivat's Conjecture in higher dimensions: if $d \geq 3$ and $n \in \mathbb{N}$, there exists aperiodic $\eta \in \{0, 1\}^{\mathbb{Z}^d}$, depending on n and d , for which $P_\eta(n, \dots, n) = 2n^{d-1} + 1$. This even rules out the possibility that an analog of Quas and Zamboni's Theorem holds for $d \geq 3$. The construction described by Sander and Tijdeman is a discretization of two skew lines in \mathbb{R}^d , and so does not provide a counterexample to Nivat's Conjecture in dimension two.

As with the Morse-Hedlund Theorem, Nivat's conjectured relation between complexity and periodicity is sharp: the aperiodic coloring $\delta \in \{0, 1\}^{\mathbb{Z}^2}$ with a 1 at $(0, 0)$ and 0 elsewhere satisfies $P_\delta(n, k) = nk + 1$ for all integers $n, k \geq 1$. In contrast to the Morse-Hedlund Theorem, the relation is not an equivalence. Berth   and Vuillon [1] and Cassaigne [6] gave examples of infinite 2-dimensional words η that are periodic, but whose block complexity satisfies $P_\eta(n, k) = 2^{n+k-1}$ for all integers $n, k \geq 1$.

Further partial results connected to Nivat's Conjecture and its generalizations are given in [1, 4, 7, 16, 15], and we refer the reader to [6] or [10] for more extensive discussions.

Our main result is an improvement on Quas and Zamboni's Theorem:

Theorem 1.2. *For $\eta \in \mathcal{A}^{\mathbb{Z}^2}$, if there exist integers $n, k \geq 1$ such that $P_\eta(n, k) \leq \frac{nk}{2}$, then η is periodic.*

Our proof is dynamical in nature: we associate a \mathbb{Z}^2 -dynamical system with η and study its subdynamics to prove the periodicity of η .

1.2. Expansive subdynamics and the conjecture. Suppose \mathcal{A} is a finite alphabet; throughout we assume that $|\mathcal{A}| > 1$. In a classical way, we endow \mathcal{A} with the discrete topology, $\mathcal{A}^{\mathbb{Z}^d}$ with the product topology, and define a \mathbb{Z}^d -action on $\mathcal{A}^{\mathbb{Z}^d}$ by $(T^{\vec{u}}\eta)(\vec{x}) := \eta(\vec{x} + \vec{u})$ for $\vec{u} \in \mathbb{Z}^d$. With respect to this topology, the maps $T^{\vec{u}}: X \rightarrow X$ are continuous. In a slight abuse, we omit the transformations $T^{\vec{u}}$ from our notation, and let $\mathcal{O}(\eta) := \{T^{\vec{u}}\eta: \vec{u} \in \mathbb{Z}^d\}$ denote the \mathbb{Z}^d -orbit of $\eta \in \mathcal{A}^{\mathbb{Z}^d}$ and write $X_\eta := \overline{\mathcal{O}(\eta)}$.

In this dynamical setup, we can rephrase periodicity. The statement that η is periodic is equivalent to saying that \mathbb{Z}^d does not act faithfully on X_η . A word η is doubly periodic if it has two non-commensurate period vectors, and for $d = 2$, the statement η is doubly periodic becomes X_η is finite.

To understand the dynamics of subactions in a \mathbb{Z}^d -dynamical system, Boyle and Lind [2] introduced the notion of expansiveness for subspaces of \mathbb{R}^d . The condition of expansiveness for a given \mathbb{Z}^d -action is open in each of the Grassmannian manifolds of \mathbb{R}^d and important dynamical quantities, such as measure-theoretic and topological directional entropy, vary in a controlled manner within each connected component of this set [2]. Boyle and Lind define a subspace $V \subseteq \mathbb{R}^d$ to be *expansive* if there exist an *expansiveness radius* $r > 0$ and an *expansiveness constant* $\delta > 0$ such that whenever $x, y \in X$ satisfy

$$d(T^{\vec{u}}x, T^{\vec{u}}y) < \delta$$

for all \vec{u} with $d(\vec{u}, V) < r$, then $x = y$. Any subspace that is not expansive is called a *nonexpansive* subspace. They showed that \mathbb{Z}^d -dynamical systems with nonexpansive subspaces are common:

Theorem 1.3 (Boyle and Lind [2]). *Let X be an infinite compact metric space with a continuous \mathbb{Z}^d -action. For each $0 \leq k < d$, there exists a k -dimensional subspace of \mathbb{R}^d that is nonexpansive.*

When restricting to $d = 2$ and the context of $X = X_\eta$, a simple corollary is that η is doubly periodic if and only if every subspace of \mathbb{R}^2 is expansive. (As throughout the paper, we mean this with respect to the \mathbb{Z}^2 -action on X_η by translation.) When there exist $n, k \in \mathbb{N}$ such that $P_\eta(n, k) \leq nk$, the connection between expansive subspaces of \mathbb{R}^2 and periodicity of η goes deeper. We show:

Theorem 1.4. *Suppose $\eta \in \mathcal{A}^{\mathbb{Z}^2}$ and $X_\eta := \overline{\mathcal{O}(\eta)}$. If there exist $n, k \in \mathbb{N}$ such that $P_\eta(n, k) \leq nk$ and there is a unique nonexpansive 1-dimensional subspace for the \mathbb{Z}^2 -action (by translation) on X_η . Then η is periodic but not doubly periodic, the unique nonexpansive line L is a rational line through the origin, and every period vector for η is contained in L .*

Thus Nivat's Conjecture reduces to:

Modified Nivat Conjecture. *If $\eta \in \mathcal{A}^{\mathbb{Z}^2}$, $X_\eta := \overline{\mathcal{O}(\eta)}$, and there exist $n, k \in \mathbb{N}$ such that $P_\eta(n, k) \leq nk$, there is at most one nonexpansive 1-dimensional subspace for the \mathbb{Z}^2 -action (by translation) on X_η .*

Under a stronger hypothesis, on the complexity, we show that this holds:

Theorem 1.5. *If $\eta \in \mathcal{A}^{\mathbb{Z}^2}$ and there exist $n, k \in \mathbb{N}$ such that $P_\eta(n, k) \leq \frac{nk}{2}$, then there is at most one nonexpansive 1-dimensional subspace for the \mathbb{Z}^2 -action (by translation) on X_η .*

Theorem 1.2 follows immediately by combining Theorems 1.4 and 1.5.

Summarizing, if $\eta \in \mathcal{A}^{\mathbb{Z}^2}$ and there exist $n, k \in \mathbb{N}$ for which $P_\eta(n, k) \leq nk$, we prove that there is a trichotomy for the \mathbb{Z}^2 -action by translation on X_η :

- (i) **No nonexpansive 1-dimensional subspaces.** In this case, Theorem 1.3 implies that η is doubly periodic.
- (ii) **A unique nonexpansive 1-dimensional subspace.** In this case, Theorem 1.4 implies that η is periodic, but not doubly periodic.
- (iii) **At least two nonexpansive 1-dimensional subspaces.** If one could show that this case can not hold, Nivat's Conjecture follows. In Theorem 1.5, we show that this case is impossible if we strengthen the hypothesis on η to the existence of $n, k \in \mathbb{N}$ such that $P_\eta(n, k) \leq \frac{nk}{2}$.

1.3. Another reformulation of the conjecture. The proofs of Theorems 1.4 and 1.5 ultimately rely on the fact that if there exist $n, k \in \mathbb{N}$ with $P_\eta(n, k) \leq nk$, then the value of $\eta_{\vec{n}}$ can be deduced from information about the value of $\eta_{\vec{m}}$ for $\vec{m} \in \mathbb{Z}^2$ that are nearby in an appropriate sense. Following Sander and Tijdeman [15], we make the following definition:

Definition 1.6. For $\mathcal{S} \subseteq \mathbb{Z}^2$, let $\mathcal{W}(\mathcal{S}, \eta) := \{(T^{\vec{u}}\eta)|_{\mathcal{S}} : \vec{u} \in \mathbb{Z}^2\}$ be the set of distinct η -colorings of \mathcal{S} (or \mathcal{S} -words) and define the η -complexity function to be

$$P_\eta(\mathcal{S}) := |\mathcal{W}(\mathcal{S}, \eta)|.$$

Note that this generalizes the definition of the complexity function $P_\eta(n, k)$ for rectangles. It is immediate that if $\mathcal{T} \subset \mathcal{S}$, then $P_\eta(\mathcal{T}) \leq P_\eta(\mathcal{S})$.

If $\mathcal{T} \subset \mathcal{S}$ and $\alpha \in \mathcal{W}(\mathcal{T}, \eta)$, we say that α *extends uniquely to an η -coloring of \mathcal{S}* if there exists a unique $\beta \in \mathcal{W}(\mathcal{S}, \eta)$ such that $\beta|_{\mathcal{T}} = \alpha$.

We define a *discrepancy function* that measures the difference between the complexity of a set and its size:

Definition 1.7. For $\eta: \mathbb{Z}^2 \rightarrow \mathcal{A}$, the η -*discrepancy function* $D_\eta(\mathcal{S})$, or just the *discrepancy function* $D(\mathcal{S})$ when η is clear from context, is defined on the set of all nonempty, finite subsets of \mathbb{Z}^2 by

$$D_\eta(\mathcal{S}) := P_\eta(\mathcal{S}) - |\mathcal{S}|.$$

The discrepancy function has the useful property (Lemma 2.1) that if $\mathcal{S} \subset \mathbb{Z}^2$ and $x \in \mathcal{S}$, then either $D_\eta(\mathcal{S} \setminus \{x\}) \leq D_\eta(\mathcal{S})$ or every η -coloring of $\mathcal{S} \setminus \{x\}$ extends uniquely to an η -coloring of \mathcal{S} . The discrepancy of any one element subset of \mathbb{Z}^2 is $|\mathcal{A}| - 1 > 0$ and the hypothesis of Nivat's conjecture is that there exists a rectangular subset of \mathbb{Z}^2 whose discrepancy is non-positive. This implies (Corollary 2.5) the existence of a set $\mathcal{S} \subset \mathbb{Z}^2$ with the property that for any $x \in \mathcal{S}$, every η -coloring of $\mathcal{S} \setminus \{x\}$ extends uniquely to an η -coloring of \mathcal{S} .

In this terminology, the Modified Nivat Conjecture becomes:

Modified Nivat Conjecture (second version). *If $\eta \in \mathcal{A}^{\mathbb{Z}^2}$, $X_\eta := \overline{\mathcal{O}(\eta)}$, and there exists an n by k rectangular subset R of \mathbb{Z}^2 satisfying $D_\eta(R) \leq 0$, then there is at most one nonexpansive 1-dimensional subspace for the \mathbb{Z}^2 -action (by translation) on X_η .*

Theorem 1.4 uses the set \mathcal{S} to show that for all $\vec{u} \in \mathbb{Z}^2$, the value of $T^{\vec{u}}\eta$ along a strip depends only on its restriction to a particular finite set. Theorem 1.5 is more subtle. The stronger hypothesis on the complexity $P_\eta(n, k)$ allows us to show that nonexpansiveness gives rise to periodicity along strips (a more precise statement is contained in Proposition 4.8). When there are multiple nonexpansive subspaces, this forces η to be doubly periodic on large, finite subsets of \mathbb{Z}^2 . We then complete the argument with an elaborate proof by contradiction by analyzing η on the boundary of these subsets.

1.4. A guide to the paper. Sections 2 and 3 develop tools for analyzing the nonexpansive subspaces of X_η . In Section 2, we define η -*generating sets*, which allow us to extend colorings to large regions, and prove a number of elementary lemmas establishing their existence and properties. In Section 3, we use this machinery to prove Theorem 1.4. Along the way, we provide a new proof of Boyle and Lind's Theorem (Theorem 1.3) adapted to our setting. The remainder of the paper is devoted to the proof of Theorem 1.5. In Section 4, we show how the stronger complexity bound assumed in Theorem 1.5 can be used to obtain additional control over the set of nonexpansive directions for X_η and in Section 5, we use the machinery from Section 4 to complete the proof of Theorem 1.5.

2. GENERATING SETS AND PERIODICITY

2.1. Geometric notation and terminology. If $R \subset \mathbb{R}^2$, we denote the convex hull of R by $\text{conv}(R)$. A subset $\mathcal{S} \subseteq \mathbb{Z}^2$ is called *convex* if $\mathcal{S} = \text{conv}(\mathcal{S}) \cap \mathbb{Z}^2$, and we let $\partial\mathcal{S}$ denote the boundary of $\text{conv}(\mathcal{S})$. A *boundary vertex* of a convex set $\mathcal{S} \subseteq \mathbb{Z}^2$ is a point in $\partial\mathcal{S} \cap \mathbb{Z}^2$ which is a vertex of the convex polygon $\partial\mathcal{S}$, and a *boundary edge* of \mathcal{S} an edge of $\partial\mathcal{S}$. We use $V(\mathcal{S})$ to denote the set of boundary vertices of \mathcal{S} and $E(\mathcal{S})$ to denote the set of boundary edges of \mathcal{S} .

If $\mathcal{S} \subset \mathbb{Z}^2$ is convex and $\text{conv}(\mathcal{S})$ has positive area, our standard convention is that the boundary of \mathcal{S} is positively oriented. When $|\mathcal{S}| < \infty$, this orientation endows each $w \in E(\mathcal{S})$ with a well-defined *successor* edge, denoted $\text{succ}(w) \in E(\mathcal{S})$ and a *predecessor* edge, denoted $\text{pred}(w) \in E(\mathcal{S})$. In the case that $|\mathcal{S}| = \infty$, there is a unique $w_\alpha \in E(\mathcal{S})$ that does not have a predecessor edge and a unique $w_\omega \in E(\mathcal{S})$ that does not have a successor edge. We extend the functions $\text{succ}(\cdot)$ and $\text{pred}(\cdot)$ to infinite convex sets in the natural way (leaving $\text{pred}(w_\alpha)$ and $\text{succ}(w_\omega)$ undefined). With these conventions, each $w \in E(\mathcal{S})$ inherits an orientation from the boundary of \mathcal{S} , and so we make a slight abuse of the notation by viewing $w \in E(\mathcal{S})$ as both a set and an oriented line segment. Thus we can refer to an oriented line in \mathbb{R}^2 as being *parallel* or *antiparallel* (or neither) to a given element of $E(\mathcal{S})$.

A convex set $H \subset \mathbb{Z}^2$ is called a *half plane* if $\text{conv}(H)$ has positive area and $E(H)$ contains only a single edge. In this case, the unique boundary edge is a line in \mathbb{R}^2 . Given $\vec{v} \in \mathbb{R}^2 \setminus \{\vec{0}\}$, a \vec{v} -half plane is a half plane whose boundary edge is parallel to \vec{v} . If $\mathcal{S} \subset \mathbb{Z}^2$ is convex and $\vec{v} \in \mathbb{R}^2 \setminus \{\vec{0}\}$, then the intersection of all \vec{v} -half planes containing \mathcal{S} is a \vec{v} -half plane whose boundary $\ell(\vec{v}, \mathcal{S})$ has nonempty intersection with $\partial\mathcal{S}$. In this case, $\ell(\vec{v}, \mathcal{S})$ is the *support line* of \mathcal{S} determined by \vec{v} . Note that $\ell(\vec{v}, \mathcal{S}) \cap \text{conv}(\mathcal{S})$ is either a boundary edge or a boundary vertex of \mathcal{S} .

We make use of two notions of size:

- (i) If $\mathcal{S} \subseteq \mathbb{Z}^2$, then $|\mathcal{S}|$ denotes the cardinality of \mathcal{S} .
- (ii) If $w \subset \mathbb{R}^2$ is a line segment, then $\|w\|$ denotes the length of w .

In particular, if $\mathcal{S} \subset \mathbb{Z}^2$ is a finite convex set and $w \in E(\mathcal{S})$, then $\|w\|$ is the length of w , while $|w \cap \mathcal{S}|$ is the number of integer points on it.

We denote the n by k rectangle based at the origin by

$$R_{n,k} := [0, n-1] \times [0, k-1].$$

2.2. The discrepancy function and η -generating sets. We use the discrepancy function to derive a number of useful properties of functions $\eta \in \mathcal{A}^{\mathbb{Z}^2}$ satisfying $P_\eta(R_{n,k}) \leq nk$ for some $n, k \in \mathbb{N}$.

Lemma 2.1. *Suppose $\mathcal{S} \subset \mathbb{Z}^2$ is finite, convex, and $|\mathcal{S}| \geq 2$. If x is a boundary vertex of \mathcal{S} , then either every η -coloring of $\mathcal{S} \setminus \{x\}$ extends uniquely to an η -coloring of \mathcal{S} or $D_\eta(\mathcal{S} \setminus \{x\}) \leq D_\eta(\mathcal{S})$.*

Proof. If there is some η -coloring of $\mathcal{S} \setminus \{x\}$ that extends non-uniquely to an η -coloring of \mathcal{S} , then $P_\eta(\mathcal{S} \setminus \{x\}) < P_\eta(\mathcal{S})$. Thus

$$D_\eta(\mathcal{S} \setminus \{x\}) = P_\eta(\mathcal{S} \setminus \{x\}) - |\mathcal{S} \setminus \{x\}| \leq (P_\eta(\mathcal{S}) - 1) - (|\mathcal{S}| - 1) = D_\eta(\mathcal{S}). \quad \square$$

Motivated by Lemma 2.1, we make the following definition:

Definition 2.2. If $\mathcal{S} \subset \mathbb{Z}^2$ is a finite, convex set and $x \in \mathcal{S}$, we say that x is η -generated by \mathcal{S} if every η -coloring of $\mathcal{S} \setminus \{x\}$ extends uniquely to an η -coloring of \mathcal{S} . A finite, nonempty, convex subset of \mathbb{Z}^2 for which every boundary vertex is generated is called an η -generating set.

Lemma 2.3. *Suppose $\mathcal{S} \subset \mathbb{Z}^2$ is a finite, convex set and $D_\eta(\mathcal{S}) \leq 0$. Let \mathcal{T} be a minimal set (with respect to partial ordering by inclusion) among all nonempty convex subsets of \mathcal{S} with discrepancy at most $D_\eta(\mathcal{S})$. Then \mathcal{T} is an η -generating set, and if $y \in V(\mathcal{T})$, $D_\eta(\mathcal{T} \setminus \{y\}) = D_\eta(\mathcal{T}) + 1$.*

Proof. We proceed by contradiction. Let $x \in \mathcal{T}$ be a boundary vertex that is not generated. Since any one element set has discrepancy $|\mathcal{A}| - 1 > 0$, \mathcal{T} must contain at least two elements; in particular $\mathcal{T} \setminus \{x\}$ is nonempty. Furthermore, $\mathcal{T} \setminus \{x\}$ is convex and by Lemma 2.1, $D_\eta(\mathcal{T} \setminus \{x\}) \leq D_\eta(\mathcal{T})$, a contradiction of the minimality of \mathcal{T} .

For the change in discrepancy, note that any vertex $y \in \mathcal{T}$ is generated and so $P_\eta(\mathcal{T}) = P_\eta(\mathcal{T} \setminus \{x\})$. The statement follows since $|\mathcal{T}| = |\mathcal{T} \setminus \{x\}| + 1$. \square

Remark 2.4. Many of our constructions assume that we have fixed a function $\eta: \mathbb{Z}^2 \rightarrow \mathcal{A}$, an η -generating set \mathcal{S} , and an edge $w \in E(\mathcal{S})$. It is often convenient to assume that the oriented line segment w points vertically downward. Such an assumption is not restrictive: if $A \in GL_2(\mathbb{Z})$, then $A^{-1}(\mathcal{S})$ is convex and

$$D_{\eta \circ A}(A^{-1}(\mathcal{S})) = D_\eta(\mathcal{S}).$$

Therefore \mathcal{S} is an η -generating set if and only if $A^{-1}(\mathcal{S})$ is an $(\eta \circ A)$ -generating set, and we have no change in the discrepancy. Since $GL_2(\mathbb{Z})$ acts transitively on directed rational lines through the origin in \mathbb{R}^2 , in constructions that only rely on \mathcal{S} being η -generating, we can always make a change of coordinates such that w is vertical with downward orientation.

Corollary 2.5. *If $\mathcal{S} \subset \mathbb{Z}^2$ is a finite, convex set with η -discrepancy $d \leq 0$, then \mathcal{S} contains a (strictly decreasing) nested family of η -generating subsets*

$$\mathcal{S}_1 \supset \dots \supset \mathcal{S}_{|d|+1}.$$

Proof. Let \mathcal{S}_1 be a nonempty, convex subset of \mathcal{S} which is minimal (with respect to inclusion) among all convex subsets having discrepancy at most d ; such a set must exist because a one element subset of \mathcal{S} has positive discrepancy. By Lemma 2.3, \mathcal{S}_1 is η -generating and contains at least two elements (because it has nonpositive η -discrepancy).

Suppose that for some $i < |d| + 1$, we have constructed η -generating sets $\mathcal{S}_1 \supset \mathcal{S}_2 \supset \dots \supset \mathcal{S}_i$ such that $D_\eta(\mathcal{S}_j) = D_\eta(\mathcal{S}) + j - 1$ for all $1 \leq j \leq i$. Let $x_i \in V(\mathcal{S}_i)$ and set $\tilde{\mathcal{S}}_i := \mathcal{S}_i \setminus \{x_i\}$. Then by Lemma 2.3, $D_\eta(\tilde{\mathcal{S}}_i) = D_\eta(\mathcal{S}_i) + 1 = D_\eta(\mathcal{S}) + i \leq 0$. We can then pass to a subset \mathcal{S}_{i+1} of $\tilde{\mathcal{S}}_i$ which is minimal among all subsets of $\tilde{\mathcal{S}}_i$ that have discrepancy at most $D_\eta(\tilde{\mathcal{S}}_i)$, and note that by Lemma 2.3, \mathcal{S}_{i+1} is η -generating. This process continues for at least $|d| + 1$ steps. \square

Corollary 2.6. *Suppose that $\mathcal{S} \subset \mathbb{Z}^2$ is a finite, convex set with η -discrepancy $d \leq 0$. For any $i \in \mathbb{N}$ and any $x_1, \dots, x_i \in \mathcal{S}$ such that $\mathcal{S} \setminus \{x_1, \dots, x_i\}$ is convex and nonempty, we have that $D_\eta(\mathcal{S} \setminus \{x_1, \dots, x_i\}) \leq D_\eta(\mathcal{S}) + i$.*

Proof. If x_j is not η -generated, then by Lemma 2.1 the discrepancy does not increase when it is removed. If x_j is η -generated, then by Lemma 2.3 the discrepancy increases by exactly one. \square

This corollary becomes relevant in our constructions: if we know that $D_\eta(\mathcal{S}) < 0$, we are free to remove any $|D_\eta(\mathcal{S})|$ elements from \mathcal{S} and are guaranteed that the resulting set contains an η -generating subset (provided it is convex).

A key fact about the discrepancy function which is crucial in Sections 4 and 5 is the following:

Lemma 2.7. *Suppose that $\mathcal{S} \subset \mathbb{Z}^2$ is a convex η -generating set for which every nonempty proper subset has strictly larger η -discrepancy and let $w \in E(\mathcal{S})$. If*

$\mathcal{S} \setminus \{w\} \neq \emptyset$, then there are at most $|w \cap \mathcal{S}| - 1$ η -colorings of $\mathcal{S} \setminus \{w\}$ that extend non-uniquely to η -colorings of \mathcal{S} .

Proof. $D_\eta(\mathcal{S} \setminus \{w\}) > D_\eta(\mathcal{S})$ (by assumption) and $|\mathcal{S} \setminus \{w\}| = |\mathcal{S}| - |w \cap \mathcal{S}|$. So, $P_\eta(\mathcal{S} \setminus \{w\}) > P_\eta(\mathcal{S}) - |w \cap \mathcal{S}|$. On the other hand, there are no more than $P_\eta(\mathcal{S}) - P_\eta(\mathcal{S} \setminus \{w\})$ distinct η -colorings of $\mathcal{S} \setminus \{w\}$ that extend non-uniquely to an η -coloring of \mathcal{S} . \square

Lemma 2.8. *Suppose that $\mathcal{S} \subset \mathbb{Z}^2$ is a finite convex set and there are two edges $w_1, w_2 \in E(\mathcal{S})$ that are antiparallel. Then any line parallel to w_1 that has nonempty intersection with \mathcal{S} contains at least $\min_{i=1,2} \{|w_i \cap \mathcal{S}| - 1\}$ integer points.*

Proof. Without loss of generality, suppose that $|w_1 \cap \mathcal{S}| \leq |w_2 \cap \mathcal{S}|$. If ℓ is a line parallel to w_1 that has nonempty intersection with \mathcal{S} , then by convexity $\|\ell \cap \text{conv}(\mathcal{S})\| \geq \|w_1\|$ (recall that $\|\cdot\|$ denotes the length of a line segment in \mathbb{R}^2). The distance between any two consecutive integer points on ℓ is the same as the distance between two consecutive integer points on the line determined by w_1 , since the two sets differ only by a translation taking the integer points on one to the integer points on the other. Since the distance between $|w_1 \cap \mathcal{S}|$ integer points on a line parallel to ℓ is exactly $\|w_1\|$, any interval in ℓ of length at least $\|w_1\|$ must contain at least $|w_1 \cap \mathcal{S}| - 1$ integer points. In particular, $\ell \cap \text{conv}(\mathcal{S})$ does. \square

We finish this subsection with two quick applications of generating sets. The first is a relation that we use in the sequel to eliminate irrational nonexpansive directions:

Lemma 2.9. *If $\eta: \mathbb{Z}^2 \rightarrow \mathcal{A}$ and there exist $n, k \in \mathbb{N}$ such that $P_\eta(n, k) \leq nk$ and $L \subset \mathbb{R}^2$ is an irrational line through the origin, then L is expansive on X_η .*

Proof. Let \mathcal{S} be an η -generating set. Choose an expansiveness radius $r > 0$ such that \mathcal{S} is contained in the set

$$U := \{\vec{u}: d(\vec{u}, L) < r\}.$$

Let ℓ be the support line of \mathcal{S} in direction L and let $\vec{w} = \mathcal{S} \cap \ell \in V(\mathcal{S})$. Define

$$c := \inf_{\vec{y} \in \mathcal{S} \setminus \{\vec{w}\}} d(\vec{y}, \ell).$$

Since \mathcal{S} is finite, $c > 0$.

We claim that $\eta|_U$ determines all of η . If not, set

$$d := \inf\{d(\vec{y}, L): \eta|_U \text{ does not determine } \eta(\vec{y})\}.$$

Then d is finite (or we are already finished) and $d \geq r$. Defining $U_R = \{\vec{u}: d(\vec{u}, L) < R\}$, then $\eta|_U$ determines $\eta|_{U_{d'}}$, where d' is either $d - c/2$, when this is positive, or $d/2$, otherwise. Choose $\vec{y} \in \mathbb{Z}^2$ such that $d(\vec{y}, L) \leq d + c/4$ and such that $\eta(\vec{y})$ is not determined by $\eta|_U$. Translating \mathcal{S} , we can assume that $\vec{w} = \vec{y}$. Then there are two possibilities. The first is that $\mathcal{S} \setminus \{\vec{y}\} \subset U_{d'}$, and then since \mathcal{S} is η -generating we have a contradiction. Otherwise, $\mathcal{S} \setminus \{\vec{w}\} \cap U_{d'} = \emptyset$, and then replacing L in the proof by its opposite orientation, the same argument leads to a contradiction. \square

The second application relates to entropy:

Definition 2.10. Suppose $\eta: \mathbb{Z}^2 \rightarrow \mathcal{A}$ and $\mathcal{S} \subset \mathbb{Z}^2$ is finite. Define

$$X_{\mathcal{S}}(\eta) := \{f: \mathbb{Z}^2 \rightarrow \mathcal{A} \text{ such that } \mathcal{W}(\mathcal{S}, f) = \mathcal{W}(\mathcal{S}, \eta)\}$$

to be the \mathbb{Z}^2 -subshift of finite type generated by the \mathcal{S} -words of η . (In more common terminology, if $F_{\mathcal{S}} := \mathcal{A}^{\mathcal{S}} \setminus \mathcal{W}(\mathcal{S}, \eta)$ is the set of all \mathcal{S} words *not* occurring in η , then $X_{\mathcal{S}}(\eta)$ is the \mathbb{Z}^2 -subshift of finite type whose set of forbidden words is $F_{\mathcal{S}}$.) An (\mathcal{S}, η) -coloring of a set $\mathcal{T} \subseteq \mathbb{Z}^2$ is any function of the form $\{f \upharpoonright \mathcal{T} : f \in X_{\mathcal{S}}(\eta)\}$.

Lemma 2.11. *If $\eta : \mathbb{Z}^2 \rightarrow \mathcal{A}$ and there is an η -generating set $\mathcal{S} \subseteq \mathbb{Z}^2$, then for any finite $\mathcal{S}' \supseteq \mathcal{S}$ the \mathbb{Z}^2 -dynamical system $(X_{\mathcal{S}'}(\eta), \{T^{\vec{u}}\}_{\vec{u} \in \mathbb{Z}^2})$ has topological entropy zero.*

Proof. Choose $n, k \in \mathbb{N}$ such that $\mathcal{S}' \subseteq R_{n,k}$. Then for any $n' > 2n$ and $k' > 2k$, the function $\eta \upharpoonright_{R_{n',k'}}$ is determined by its restriction to the set

$$[0, n' - 1] \times [0, k' - 1] \setminus [n, n' - n] \times [k, k' - k].$$

So $P_{\eta}(R_{n',k'}) \leq |\mathcal{A}|^{2nk' + 2kn' - 4nk}$ and

$$\lim_{n' \rightarrow \infty} \frac{1}{(n')^2} \log P_{\eta}(R_{n',k'}) = 0.$$

□

Remark 2.12. In dimension one, the analog of Lemma 2.11 holds and leads to (another) proof of the Morse-Hedlund theorem: a one-dimensional subshift of finite type either has positive entropy or every element is periodic. In dimension two, there are zero entropy \mathbb{Z}^2 -subshifts of finite type that do not contain any periodic elements. Thus Lemma 2.11 serves as an indication that generating sets are dynamically interesting, rather than providing an approach to Nivat's conjecture.

2.3. Ambiguous half planes and periodicity. In this section, we develop a relationship between the notions of nonexpansivity and periodicity. The main result is Lemma 2.18. To state and prove the lemma, we start with some terminology.

Definition 2.13 (Ambiguous extension). If $\mathcal{S} \subset \mathbb{Z}^2$, $\mathcal{T}_1 \subset \mathcal{T}_2 \subseteq \mathbb{Z}^2$, then a coloring $f \in X_{\mathcal{S}}(\eta)$ is $(\mathcal{S}, \mathcal{T}_1, \mathcal{T}_2, \eta)$ -ambiguous if there exist $g_1, g_2 \in X_{\mathcal{S}}(\eta)$ such that $g_1 \upharpoonright_{\mathcal{T}_1} = g_2 \upharpoonright_{\mathcal{T}_1} = f$ but $g_1 \upharpoonright_{\mathcal{T}_2} \neq g_2 \upharpoonright_{\mathcal{T}_2}$.

Ambiguity becomes especially interesting when $\mathcal{T}_2 \supset \mathcal{T}_1$ is produced in some way by \mathcal{T}_1 , and this is captured in the following definition:

Definition 2.14 (Enveloping set). If $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{Z}^2 \setminus \{\vec{0}\}$ is a collection of vectors, we say that a convex set $\mathcal{T} \subseteq \mathbb{Z}^2$ is $\{\vec{v}_1, \dots, \vec{v}_n\}$ -enveloped if for every $w \in E(\mathcal{T})$, there exists $i \in \{1, \dots, n\}$ such that w is parallel to \vec{v}_i . An *enveloping set* for \mathcal{T} is a set of vectors that envelops it. A *minimal enveloping set* for \mathcal{T} is a collection of vectors that envelops \mathcal{T} and such that no proper subset suffices.

Given a convex region, we define an extension over an edge of the region (it may help to refer to Figure 1 while reading this definition). The definition splits into several cases depending on the type of edge:

Definition 2.15. Suppose $\mathcal{T} \subseteq \mathbb{Z}^2$ is convex, $\text{conv}(\mathcal{T})$ has positive area, and each $w \in E(\mathcal{T})$ determines a rational line in \mathbb{R}^2 .

- (i) Suppose w points vertically downward. Without loss of generality, assume it is a subset of the y -axis.

- (a) If w has both a successor edge and a predecessor edge in $E(\mathcal{T})$, choose $a, b, c, d \in \mathbb{Q}$ such that $\text{pred}(w) \subseteq \{(x, y) \in \mathbb{R}^2 : y = ax + b\}$ and $\text{succ}(w) \subseteq \{(x, y) \in \mathbb{R}^2 : y = cx + d\}$. If there is an integer $\Delta < 0$ such that $c\Delta + d \leq a\Delta + b$ and $c\Delta + d, a\Delta + b \in \mathbb{Z}$, then for maximal such Δ , we define the w -extension $\text{Ext}_w(\mathcal{T})$ of \mathcal{T} to be the set

$$\text{Ext}_w(\mathcal{T}) := \mathcal{T} \cup \{(x, y) \in \mathbb{Z}^2 : cx + d \leq y \leq ax + b, \Delta \leq x < 0\}.$$

If no such Δ exists, we define $\text{Ext}_w(\mathcal{T}) := \mathcal{T}$.

- (b) If w has a successor edge but does not have a predecessor edge, choose $c, d \in \mathbb{Q}$ such that $\text{succ}(w) \subseteq \{(x, y) \in \mathbb{R}^2 : y = cx + d\}$. Choose maximal $\Delta < 0$ such that $c\Delta + d \in \mathbb{Z}$. Then we define the w -extension $\text{Ext}_w(\mathcal{T})$ of \mathcal{T} to be

$$\text{Ext}_w(\mathcal{T}) := \mathcal{T} \cup \{(x, y) \in \mathbb{Z}^2 : cx + d \leq y, \Delta \leq x < 0\}.$$

- (c) If w has a predecessor edge but does not have a successor edge, choose $a, b \in \mathbb{Q}$ such that $\text{pred}(w) \subseteq \{(x, y) \in \mathbb{R}^2 : y = ax + b\}$. Choose maximal $\Delta < 0$ such that $a\Delta + b \in \mathbb{Z}$. Then we define the w -extension $\text{Ext}_w(\mathcal{T})$ of \mathcal{T} to be the set

$$\text{Ext}_w(\mathcal{T}) := \mathcal{T} \cup \{(x, y) \in \mathbb{Z}^2 : y \leq ax + b, \Delta \leq x < 0\}.$$

- (ii) If w does not point vertically downward, let $A \in GL_2(\mathbb{Z})$ be such that Aw points vertically downward. We define the w -extension of \mathcal{T} to be $A^{-1}(\text{Ext}_w(A\mathcal{T}))$. (Note that this set does not depend on the choice of A .)

It follows that $\text{Ext}_w(\mathcal{T})$ is a convex, B -enveloped set containing \mathcal{T} , and may be \mathcal{T} itself. If $\text{Ext}_w(\mathcal{T})$ strictly contains \mathcal{T} , then there is a finite collection of lines ℓ_1, \dots, ℓ_m such that

- ℓ_i is parallel to w for all i ;
- $\ell_i \cap (\text{Ext}_w(\mathcal{T}) \setminus \mathcal{T})$ is convex for all i ;
- we can decompose $\text{Ext}_w(\mathcal{T}) \setminus \mathcal{T}$ as the disjoint union:

$$\text{Ext}_w(\mathcal{T}) \setminus \mathcal{T} = \bigsqcup_{i=1}^m (\ell_i \cap \text{Ext}_w(\mathcal{T})).$$

In this case, m is called the *depth* of the extension. For $1 \leq j \leq m$,

$$\mathcal{T} \cup \bigcup_{i=1}^j (\ell_i \cap \text{Ext}_w(\mathcal{T}))$$

is the (w, j) -subextension of \mathcal{T} (note that the (w, j) -subextension may not be B -enveloped). The $(w, 0)$ -subextension of \mathcal{T} is \mathcal{T} itself.

Definition 2.16. Suppose $\mathcal{S}, \mathcal{T} \subset \mathbb{Z}^2$ are convex and $w \in E(\mathcal{T})$. If $\text{Ext}_w(\mathcal{T}) \neq \mathcal{T}$ and if $f \in X_{\mathcal{S}}(\eta)$, we say that $f|_{\mathcal{T}}$ is (\mathcal{S}, w, η) -ambiguous if it is $(\mathcal{T}, \text{Ext}_w(\mathcal{T}), \eta)$ -ambiguous.

Suppose H is a half plane and $w \in E(H)$ is its unique boundary edge. If $f \in X_{\mathcal{S}}(\eta)$, we say that $f|_H$ is (\mathcal{S}, η) -ambiguous if it is (\mathcal{S}, w, η) -ambiguous.

Definition 2.17. If \mathcal{T} is a convex set, $\mathcal{S} \subset \mathcal{T}$ is convex, and $w \in E(\mathcal{S})$ is parallel to an edge $\hat{w} \in E(\mathcal{T})$, let $V_{\mathcal{S}, \mathcal{T}, w}$ be the set of translations that take \mathcal{S} to a subset of \mathcal{T} such that the edges parallel to w overlap:

$$V_{\mathcal{S}, \mathcal{T}, w} = \{\vec{v} \in \mathbb{Z}^2 : (\mathcal{S} + \vec{v}) \subseteq \mathcal{T}, (w + \vec{v}) \subseteq \hat{w}\}.$$

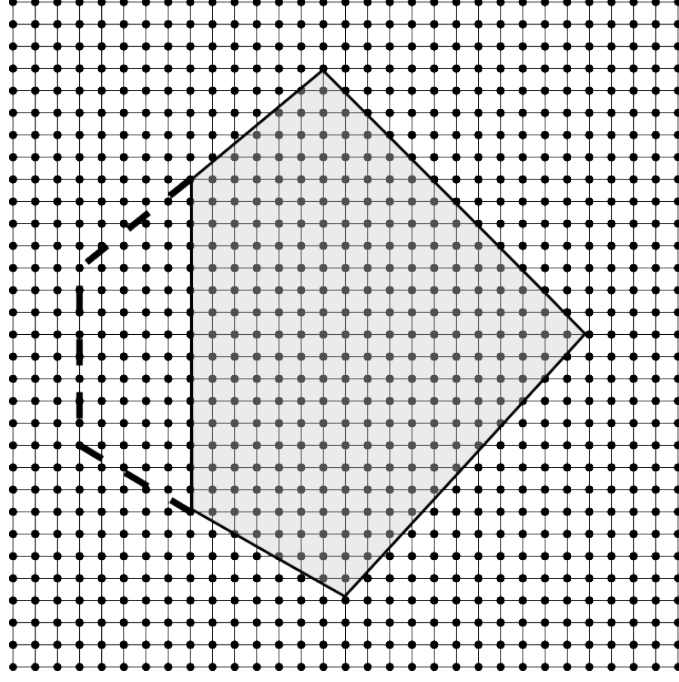


FIGURE 1. \mathcal{T} is the set of integer points enclosed by the solid lines. If w points vertically downward, then $\text{Ext}_w(\mathcal{T})$ is enclosed by the dashed lines and the nonvertical solid lines. The set $\text{Ext}_w(\mathcal{T}) \setminus \mathcal{T}$ decomposes into six vertically aligned sets which determine the subextensions. The depth of the extension is six.

There exist vectors $\vec{a}_{\mathcal{S}, \mathcal{T}, w}, \vec{b}_{\mathcal{S}, \mathcal{T}, w} \in \mathbb{Z}^2$ such that

$$V_{\mathcal{S}, \mathcal{T}, w} = \{\vec{a}_{\mathcal{S}, \mathcal{T}, w} + \lambda \vec{b}_{\mathcal{S}, \mathcal{T}, w} : \lambda \in I\},$$

where

$$I = \begin{cases} \{0, 1, \dots, |V_{\mathcal{S}, \mathcal{T}, w}| - 1\} & \text{if } \|\hat{w}\| < \infty; \\ \mathbb{N} \cup \{0\} & \text{if } \hat{w} \text{ is a semi-infinite line;} \\ \mathbb{Z} & \text{if } \hat{w} \text{ is a line.} \end{cases}$$

Let I_{\min}, I_{\max} be the minimum and maximum elements of I , respectively (allowing $I_{\min} = -\infty$ and $I_{\max} = +\infty$ if necessary). Then the (\mathcal{S}, w) -border of \mathcal{T} is the set

$$\bigcup_{\vec{v} \in V_{\mathcal{S}, \mathcal{T}, w}} (\mathcal{S} + \vec{v}).$$

For $g \in \mathbb{N}$, the g -interior of the (\mathcal{S}, w) -border is the set

$$\bigcup_{\lambda = I_{\min} + g}^{I_{\max} - g - 1} (\mathcal{S} + \vec{a}_{\mathcal{S}, \mathcal{T}, w} + \lambda \vec{b}_{\mathcal{S}, \mathcal{T}, w}).$$

We show that ambiguity is a source of periodicity:

Lemma 2.18. *Suppose $\eta: \mathbb{Z}^2 \rightarrow \mathcal{A}$, $\mathcal{S} \subset \mathbb{Z}^2$ is an η -generating set and there exist antiparallel $w_1, w_2 \in E(\mathcal{S})$. Suppose $|w_1| \leq |w_2|$, H is a w_1 -half plane, and the*

restriction of $f \in X_{\mathcal{S}}(\eta)$ to H is (\mathcal{S}, η) -ambiguous. Then the $(\mathcal{S} \setminus \{w_1\}, w_1)$ -border of H is periodic with period vector parallel to w_1 . Its period is at most $|w_1 \cap \mathcal{S}| - 1$.

Proof. Without loss of generality (see Remark 2.4), we assume that w_1 and w_2 are vertical, and w_1 points downward. Let $h := |w_1 \cap \mathcal{S}| - 1$ and for each vertical line ℓ with nonempty intersection with \mathcal{S} , let \vec{x}_ℓ denote the bottom-most element of $\ell \cap \mathcal{S}$. By Lemma 2.8,

$$\mathcal{T} := \bigcup_{i=0}^{h-1} \{\vec{x}_\ell + (0, i) \in \mathbb{Z}^2 : \ell \text{ vertical}, \ell \cap \mathcal{S} \neq \emptyset\} \subseteq \mathcal{S}.$$

Define $\tilde{\mathcal{S}} := \mathcal{S} \setminus \{w_1\}$, $\tilde{\mathcal{T}} := \mathcal{T} \setminus \{w_1\}$, and fix a vector $\vec{u} \in \mathbb{Z}^2$ such that $\tilde{\mathcal{T}} + \vec{u}$ is contained in the $(\tilde{\mathcal{S}}, w_1)$ -border of H .

We claim that for any $\lambda \in \mathbb{Z}$, the η -coloring of $\tilde{\mathcal{S}}$ given by $f|_{\tilde{\mathcal{S}}} + \vec{u} + (0, \lambda)$ has at least two extensions to an η -coloring of \mathcal{S} . Instead, suppose not. Then the coloring $f|_H$ uniquely determines the η -coloring of $H \cup \{\mathcal{S} + \vec{u} + (0, \lambda)\}$, which in particular determines the η -coloring of all but one of the elements of the set $\mathcal{S} + \vec{u} + (0, \lambda + 1)$. Since \mathcal{S} is η -generating, this uniquely determines the η -coloring of $H \cup (\mathcal{S} + \vec{u} + (0, \lambda + 1))$. Now for $i \geq 0$, suppose the η -coloring of

$$H \cup \{\mathcal{S} + \vec{u} + (0, \lambda + j) : 0 \leq j \leq i\}$$

has been determined. Then the η -coloring of all but one of the elements of the set $\mathcal{S} + \vec{u} + (0, \lambda + i + 1)$ is determined. Since \mathcal{S} is η -generating, this determines the η -coloring of $H \cup \{\mathcal{S} + \vec{u} + (0, \lambda + i + 1)\}$. By induction, this holds for all $i \geq 0$. Similarly for all $i \leq 0$. But this contradicts the ambiguity of the η -coloring of H .

Recall that since \mathcal{S} is η -generating, $D_\eta(\tilde{\mathcal{S}}) > D_\eta(\mathcal{S})$ and so $P_\eta(\tilde{\mathcal{S}}) > P_\eta(\mathcal{S}) - |w_1 \cap \mathcal{S}|$. Therefore, the number of η -colorings of $\tilde{\mathcal{S}}$ that do not uniquely extend η -colorings of \mathcal{S} is at most $h = |w_1 \cap \mathcal{S}| - 1$. In particular, there are at most h η -colorings of $\tilde{\mathcal{T}}$ that arise as the restriction of f to a set of the form $\tilde{\mathcal{T}} + \vec{u} + (0, i)$, where $i \in \mathbb{Z}$.

Define a color set $\tilde{\mathcal{A}}$ whose colors are η -colorings of $\{\vec{x}_\ell : \ell \text{ is vertical}, \ell \cap \tilde{\mathcal{S}} \neq \emptyset\}$ occurring as the restriction of f to a set of the form

$$B_i := \{\vec{x}_\ell : \ell \text{ is vertical}, \ell \cap \tilde{\mathcal{S}} \neq \emptyset\} + \vec{u} + (0, i),$$

where $i \in \mathbb{Z}$. Define $g : \mathbb{Z} \rightarrow \tilde{\mathcal{A}}$ by $g(i) = f|_{B_i}$. Then the (one-dimensional) block complexity $P_g(h)$ is the number of η -colorings of $\tilde{\mathcal{T}}$ occurring as the restriction of f to a set of the form $\tilde{\mathcal{T}} + \vec{u} + (0, i)$. But we have shown that $P_g(h) \leq h$. By the Morse-Hedlund Theorem, g is periodic with period at most h . The result now follows from the definition of g . \square

The proof of Lemma 2.18 holds in a slightly more general setting, and we make use of these forms in Section 4. Corollary 2.19 generalizes the result to the case of a more general convex set instead of a half plane, and Corollary 2.20 replaces the assumption that \mathcal{S} is η -generating with a more general condition.

Corollary 2.19. *Suppose $\eta : \mathbb{Z}^2 \rightarrow \mathcal{A}$, $\mathcal{S} \subset \mathbb{Z}^2$ is an η -generating set, and there are antiparallel $w_1, w_2 \in E(\mathcal{S})$. Suppose $|w_1 \cap \mathcal{S}| \leq |w_2 \cap \mathcal{S}|$, \mathcal{T} is a convex set containing \mathcal{S} , and there is some $\hat{w}_1 \in E(\mathcal{T})$ that is parallel to w_1 and sufficiently long such that $\text{Ext}_{\hat{w}_1}(\mathcal{T}) \neq \mathcal{T}$.*

If $f \in X_S(\eta)$ and $f|_{\mathcal{T}}$ is $(\mathcal{T}, \hat{w}, \eta)$ -ambiguous, then there is some j , between 0 and the depth of the \hat{w} -extension of \mathcal{T} , such that the restriction of f to the $|w_1 \cap \mathcal{S}|$ -interior of the $(\mathcal{S} \setminus \{w_1\}, \hat{w}_1)$ -border of the (w, j) -subextension of \mathcal{T} is periodic with period vector parallel to w_1 and of period at most $|w_1 \cap \mathcal{S}| - 1$.

Proof. Choose the integer j to be the index of the largest (w, j) -subextension of \mathcal{T} to which the η -coloring of \mathcal{T} , given by $f|_{\mathcal{T}}$, extends uniquely. Thereafter, the proof is identical to that of Lemma 2.18, except that the application of the Morse-Hedlund Theorem is to a finite (or semi-infinite) interval in \mathbb{Z} instead of to \mathbb{Z} itself. \square

Corollary 2.20. Suppose $\eta: \mathbb{Z}^2 \rightarrow \mathcal{A}$ and $\mathcal{S} \subset \mathbb{Z}^2$ is a convex set for which

- There exists $w \in E(\mathcal{S})$ such that for any line ℓ parallel to w that has nonempty intersection with \mathcal{S} , we have $|\ell \cap \mathcal{S}| \geq |w \cap \mathcal{S}| - 1$;
- The two endpoints of w are η -generated by \mathcal{S} ;
- $D_\eta(\mathcal{S} \setminus \{w\}) > D_\eta(\mathcal{S})$.

Then under the hypotheses of Corollary 2.19, with the assumption that \mathcal{S} is η -generating replaced by the conditions listed above, the conclusion of Corollary 2.19 holds.

Proof. The proof is the same as that of Lemma 2.18, as these assumptions on \mathcal{S} are the only ones that were used. \square

3. RIGIDITY, SHIFTABILITY, AND PERIODICITY

3.1. Rigid directions. Suppose that $\eta: \mathbb{Z}^2 \rightarrow \mathcal{A}$ satisfies $D_\eta(R_{n,k}) \leq 0$ for some $n, k \in \mathbb{N}$. By Corollary 2.5, there is an η -generating subset $\mathcal{S} \subseteq R_{n,k}$. A half plane cannot have an (\mathcal{S}, η) -ambiguous coloring unless the unique boundary edge is parallel to an edge of \mathcal{S} , as otherwise we can use \mathcal{S} to extend the coloring uniquely to a larger half-plane. However \mathcal{S} may not be uniquely determined (and by Corollary 2.5 it is not unique if the discrepancy is strictly negative) and so a \vec{v} -half plane cannot have an (\mathcal{S}, η) -ambiguous coloring unless *every* η -generating subset of $R_{n,k}$ has an edge parallel to \vec{v} . This motivates the following definition:

Definition 3.1. Suppose that $\eta: \mathbb{Z}^2 \rightarrow \mathcal{A}$, $\ell \subset \mathbb{R}^2$ is a rational line through the origin and $A \in GL_2(\mathbb{Z})$ maps the y -axis to ℓ . For $a, b \in \mathbb{N}$, we say that ℓ is η -shiftable with parameters (a, b, A) if every $(\eta \circ A)$ -coloring of $[0, a] \times [-b, b]$ extends uniquely to an $(\eta \circ A)$ -coloring of $[0, a] \times [-b, b] \cup \{(-1, 0)\}$. A rational line is η -shiftable if there exist $a, b \in \mathbb{N}$ and $A \in GL_2(\mathbb{Z})$ such that it is η -shiftable with parameters (a, b, A) , or just shiftable when η is clear from the context.

A rational line through the origin which is not shiftable is called η -rigid, or just rigid when η is clear from the context. We can also refer to a vector $\vec{v} \in \mathbb{Z}^2 \setminus \{\vec{0}\}$ as being *shiftable* or *rigid*, meaning that the span of \vec{v} is shiftable or rigid, respectively.

Although the y -axis seems to play a distinguished role in the definitions of shiftable and rigid, the choice of this direction is arbitrary (see Remark 2.4).

The notion of shiftability, and more importantly rigidity, are one sided versions of expansiveness and nonexpasiveness used in Boyle and Lind [2]. More precisely, shiftability is a property of a directed line, while expansiveness is a property of a line; a line is expansive when both possible orientations of the line are shiftable. They play a similar role, allowing an extension of given information in a region to a larger region.

Lemma 3.2. *Suppose $\eta: \mathbb{Z}^2 \rightarrow \mathcal{A}$ and $D_\eta(R_{n,k}) \leq 0$ for some $n, k \in \mathbb{N}$. If $\mathcal{S} \subseteq R_{n,k}$ is an η -generating set, then a rational line ℓ through the origin is η -rigid if and only if there exists an (\mathcal{S}, η) -ambiguous ℓ -half plane P .*

Moreover, there exist $f, g \in \overline{\mathcal{O}(\eta)}$ such that the restrictions of f and g to the half plane P coincide, but they differ on its ℓ -extension.

Proof. If there is an (\mathcal{S}, η) -ambiguous ℓ -half plane, then ℓ must be rigid; otherwise ℓ -expansivity and the fact that \mathcal{S} is a generating set contradicts ambiguity of the half plane. Conversely, if ℓ is rigid and $A \in GL_2(\mathbb{Z})$ maps the y -axis to ℓ , then for every $a, b \in \mathbb{N}$ there is an $(\eta \circ A)$ -coloring of $[0, a] \times [-b, b]$ that has two extensions to an $(\eta \circ A)$ -coloring of $[0, a] \times [-b, b] \cup \{(-1, 0)\}$. Let $f_a, g_a \in \mathcal{O}(\eta \circ A)$ be two such extensions of $[0, a] \times [-a, a]$. By compactness, there exist accumulation points $f, g \in \overline{\mathcal{O}(\eta \circ A)}$ for the sequences (f_n) and (g_n) , respectively. Then $f|_{H_0} = g|_{H_0}$, but $f|_{H_{-1}} \neq g|_{H_{-1}}$. Applying A^{-1} to this half plane gives the result. \square

The following simple lemma is used to limit possible directions of periodicity:

Lemma 3.3. *Suppose $\eta: \mathbb{Z}^2 \rightarrow \mathcal{A}$ and there exist $n, k \in \mathbb{N}$ such that $P_\eta(R_{n,k}) \leq nk$. If \mathcal{S} is an η -generating set, then for any η -rigid direction ℓ , there is a boundary edge $w_\ell \in E(\mathcal{S})$ parallel to ℓ .*

In particular, ℓ can be translated such that it intersects $R_{n,k}$ in at least two places.

Proof. Suppose $\mathcal{S} \subseteq R_{n,k}$ is an η -generating set but any translation of ℓ intersects \mathcal{S} in at most one place. Choose a translation of ℓ which intersects \mathcal{S} at a vertex, and without loss of generality assume this translation of ℓ intersects \mathcal{S} at the origin. Let $A \in GL_2(\mathbb{Z})$ be a map taking the y -axis to ℓ . Choose an $(A^{-1}(\mathcal{S}), \eta \circ A)$ -ambiguous coloring of $H_0 := \{(x, y) \in \mathbb{Z}^2: x \geq 0\}$. Notice that

$$A^{-1}(\mathcal{S}) \cap \{(0, y): y \in \mathbb{Z}\} = (0, 0).$$

Since $(0, 0) \in A^{-1}(\mathcal{S})$ is $\eta \circ A$ -generated, there is a unique extension of any η -coloring of H_0 to an η -coloring of $\{(x, y) \in \mathbb{Z}^2: x \geq -1\}$. This contradicts the $(A^{-1}(\mathcal{S}), \eta \circ A)$ -ambiguity of the coloring of H_0 . \square

Combining this lemma with Lemma 2.9, we have:

Corollary 3.4. *Suppose $\eta: \mathbb{Z}^2 \rightarrow \mathcal{A}$ and there exist $n, k \in \mathbb{N}$ such that $P_\eta(R_{n,k}) \leq nk$. If L is a nonexpansive lines for X_η , then there exists a translation of L that intersects $R_{n,k} \cap \mathbb{Z}^2$ in at least two points.*

3.2. A characterization of double periodicity.

Lemma 3.5. *Suppose $\vec{v}_1, \dots, \vec{v}_m \in \mathbb{Z}^2 \setminus \{\vec{0}\}$. Given $n \in \mathbb{N}$, there exists $A = A(n, \vec{v}_1, \dots, \vec{v}_m) \in \mathbb{N}$ such that any finite, convex $\mathcal{S} \subset \mathbb{Z}^2$ containing at least A integer points and such that $\partial\mathcal{S}$ is $(\vec{v}_1, \dots, \vec{v}_m)$ -enveloped, has a boundary edge that contains at least n integer points.*

Proof. For each $i = 1, 2, \dots, m$ choose a length $L_i \in \mathbb{R}$ such that any rational line parallel to \vec{v}_i of length at least L_i contains at least n integer points. Define

$$A := \left\lceil \frac{(L_1 + \dots + L_m)^2}{4\pi} \right\rceil + mn.$$

By Pick's Theorem, the area of $\text{conv}(\mathcal{S})$ is given by

$$(\# \text{ of integer points inside } \text{conv}(\mathcal{S})) + \frac{(\# \text{ of integer points on } \partial\mathcal{S})}{2} - 1.$$

Since \mathcal{S} contains at least A integer points, either the number of integer points on $\partial\mathcal{S}$ is at least mn or the area of $\text{conv}(\mathcal{S}) \geq \frac{1}{4\pi}(L_1 + \dots + L_m)^2$. In the former case, at least one of the edges of $\partial\mathcal{S}$ contains n integer points. In the latter case, the isoperimetric inequality implies that the length of $\partial\mathcal{S}$ is at least $L_1 + \dots + L_m$, and so at least one of the edges contains n integer points. \square

Lemma 3.6. *Suppose $\eta: \mathbb{Z}^2 \rightarrow \mathcal{A}$ and $\mathcal{S} \subset \mathbb{Z}^2$ is a finite, convex set whose boundary edges are labeled w_1, \dots, w_n where $w_{i+1} = \text{succ}(w_i)$ for all $i = 1, \dots, n$ (indices are taken mod n). Assume that there exist $a, b, c, d \in \mathbb{Q}$ such that*

- $a \geq 0, c \leq 0, d \geq b$;
- w_1 is vertical;
- w_n is parallel to the line $y = ax + b$;
- w_2 is parallel to the line $y = cx + d$.

If the two endpoints of w_1 are η -generated by \mathcal{S} , then any (η, \mathcal{S}) -coloring of the region

$$\mathcal{R} := \{(x, y) \in \mathbb{Z}^2 : cx + d \leq y \leq ax + b, x \geq 0\}$$

and of any $|w_1 \cap \mathcal{S}| - 1$ consecutive integer points of the line segment $\{(-1, y) \in \mathbb{Z}^2 : d \leq y \leq b\}$ extends uniquely to an (\mathcal{S}, η) -coloring of

$$\{(x, y) \in \mathbb{Z}^2 : cx + d \leq y \leq ax + b, x \geq -1\}.$$

It is important to note we make no assumption that the lines $y = ax + b$ or $y = cx + d$ intersect the line $x = -1$ at an integer point.

Proof. Let $h := |w_1 \cap \mathcal{S}| - 1$. Suppose we know the η -coloring of the points $(-1, y - h + 1), \dots, (-1, y)$ for some $y \leq b - a - 1$. Let $\vec{v} \in \mathbb{Z}^2$ be the translation that takes the top most element of w_1 to the point $(-1, y)$. Since $y + 1 \leq b - a$, the line through $(-1, y + 1)$ parallel to w_n is in the region $\{y \leq ax + b\}$. Since \mathcal{S} is convex and w_n is parallel to $y = ax + b$, we have that $\mathcal{S} + \vec{v}$ lies in the region $\mathcal{R} \cup \{(-1, y - h + 1), \dots, (-1, y + 1)\}$ (see Figure 2).

Since the endpoints of w_1 are η -generated by \mathcal{S} , the color at $\eta(-1, y + 1)$ can be determined by the restriction of η to the rest of the elements of $\mathcal{S} + \vec{v}$. Continuing inductively, the (\mathcal{S}, η) -coloring of the region $\mathcal{R} \cup \{(-1, y - h + 1), \dots, (-1, y + 1)\}$ extends uniquely to an (\mathcal{S}, η) -coloring of $\mathcal{R} \cup \{(-1, y - h + 1), \dots, (-1, \lfloor b - a \rfloor)\}$.

A similar argument shows we can uniquely extend the (\mathcal{S}, η) -coloring of the region $\mathcal{R} \cup \{(-1, y - h + 1), \dots, (-1, \lfloor b - a \rfloor)\}$ to an (\mathcal{S}, η) -coloring of the region $\mathcal{R} \cup \{(-1, \lceil d - c \rceil), \dots, (-1, \lfloor b - a \rfloor)\}$. \square

Corollary 3.7. *Assume that $\eta: \mathbb{Z}^2 \rightarrow \mathcal{A}$ and $\mathcal{S} \subset \mathbb{Z}^2$ is a finite, convex set whose boundary edges are labeled w_1, \dots, w_n where $w_{i+1} = \text{succ}(w_i)$ for $i = 1, \dots, n$ (indices are taken mod n). Suppose that \mathcal{T} is a convex, $\{w_1, \dots, w_n\}$ -enveloped set and that $\tilde{\mathcal{T}}$ is the w_1 -extension of \mathcal{T} . Then any (η, \mathcal{S}) -coloring of \mathcal{T} and of any $|w_1 \cap \mathcal{S}| - 1$ consecutive integer points on each line parallel to w_1 that has non-empty intersection with $\tilde{\mathcal{T}} \setminus \mathcal{T}$ extends uniquely to an η -coloring of $\tilde{\mathcal{T}}$.*

Proof. After a linear change of coordinates mapping w_1 to the vertical direction, this follows by repeated applications of Lemma 3.6. \square

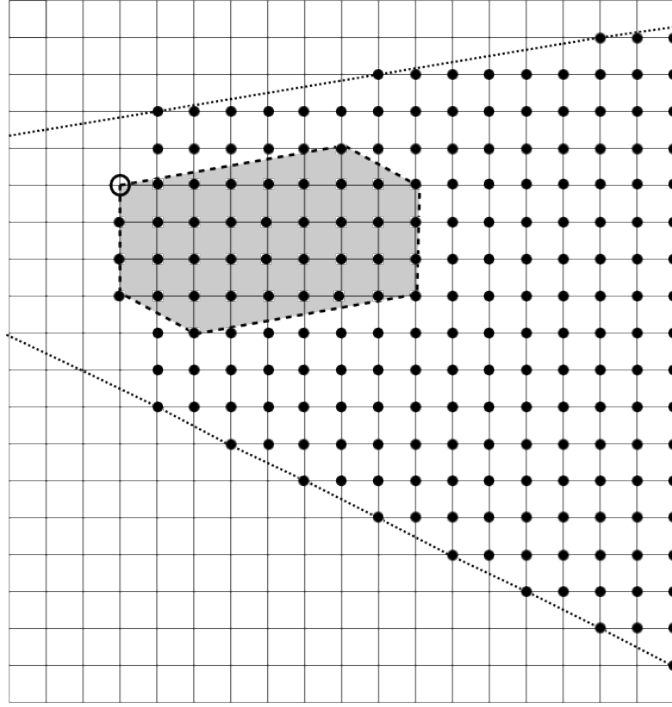


FIGURE 2. The dotted points in \mathbb{Z}^2 denote the region on which the coloring is known. The color of the topmost element of $w_1 + \vec{v}$ (denoted by the open circle) can be deduced from the coloring of the rest of $\mathcal{S} + \vec{v}$.

This leads us to necessary and sufficient conditions for double periodicity, a result that can be derived from Boyle and Lind's Theorem (Theorem 1.3). We include a complete proof, as we need further information that can be derived from the finer notion of shiftability, as opposed to expansiveness. In particular, techniques of the proof are also used to understand the case of a unique direction of expansivity (Theorem 1.4).

Theorem 3.8. *The coloring $\eta: \mathbb{Z}^2 \rightarrow \mathcal{A}$ is doubly-periodic if and only if there exist $n, k \in \mathbb{N}$ such that $P_\eta(R_{n,k}) \leq nk$ and η has no rigid directions.*

Proof. Assume that η is doubly periodic and assume that it has vertical period n and horizontal period k . Then for every $a, b \in \mathbb{N}$, $P_\eta(R_{a,b}) \leq nk$. In particular, $D_\eta(R_{1,nk}) \leq 0$ and $D_\eta(R_{nk,1}) \leq 0$. By Corollary 2.5, $R_{1,nk}$ contains an η -generating set \mathcal{S} and $R_{nk,1}$ contains an η -generating set \mathcal{T} . If ℓ is any rational line, then at least one of \mathcal{S} and \mathcal{T} is not parallel to ℓ . Since \mathcal{S} and \mathcal{T} are generating, ℓ is shifttable.

Conversely, suppose that $D_\eta(R_{n,k}) \leq 0$ and η has no rigid directions. Fix an η -generating set $\mathcal{S} \subseteq R_{n,k}$ and enumerate the edges of $\partial\mathcal{S}$ as w_1, \dots, w_m where $w_{i+1} = \text{succ}(w_i)$ for all i (indices are taken mod m).

Since none of the lines determined by w_1, \dots, w_m are rigid, we can choose parameters $a_1, \dots, a_m, b_1, \dots, b_m \in \mathbb{N}$ and $A_1, \dots, A_m \in GL_2(\mathbb{Z})$ such that the line

determined by w_i is (\mathcal{S}, η) -shiftable with parameters (a_i, b_i, A_i) . For i, j such that w_i and w_j are neither parallel nor antiparallel, let $\theta_{i,j} \in (-\pi, \pi) \setminus \{0\}$ be the angle between w_i and w_j and let c_i be the length of the orthogonal projection of $A_i^{-1}(1, 0)$ onto the direction determined by w_i . Let

$$N = \sum_{i=1}^m (2b_i + a_i c_i) + \max_{i,j} \left\{ \left| \frac{a_1 + \dots + a_m}{\tan \theta_{i,j}} \right| \right\} + \sum_{i=1}^m |w_i \cap \mathcal{S}|.$$

Any triangle formed by lines parallel to the edges of \mathcal{S} that has an edge w_i of length at least N must have a non-empty (B, w_i) -border, where

$$B = A_i^{-1}([0, a_i + |w_i \cap \mathcal{S}| - 1] \times [-b_i, b_i]).$$

In particular, any η -coloring of this triangle uniquely determines the color of at least $|w_i \cap \mathcal{S}| - 1$ consecutive integer points on its w_1 -extension (the choice if starting with w_1 is arbitrary and we could have begun with any other edge). Now suppose \mathcal{T} is convex, $\{w_1, \dots, w_m\}$ -enveloped, and one of its boundary edges is parallel to w_1 . If this edge contains at least N integer points, then by Corollary 3.7, any η -coloring of \mathcal{T} extends uniquely to an η -coloring of its w_1 -extension.

By Lemma 3.5, there exists $a \in \mathbb{N}$ such that any convex, $\{w_1, \dots, w_n\}$ -enveloped set that contains at least a integer points has a boundary edge that contains at least N integer points. Let \mathcal{T}_1^0 be such a set. Choose a boundary edge w^1 that contains at least N integer points. Repeating the same argument as in the preceding paragraph, the $\eta|_{\mathcal{T}_1^0}$ extends uniquely to the w^1 -extension of \mathcal{T}_1^0 . Call this extension \mathcal{T}_2^0 . Inductively, we construct sets

$$\mathcal{T}_1^0 \subset \mathcal{T}_2^0 \subset \dots$$

and edges $w^j \in E(\mathcal{T}_j^0)$ such that w^j contains at least N integer points and \mathcal{T}_{j+1}^0 is the w^j -extension of \mathcal{T}_j^0 . Again, applying the same argument, any η -coloring of \mathcal{T}_j^0 extends uniquely to an η -coloring of \mathcal{T}_{j+1}^0 and hence to $\bigcup_j \mathcal{T}_j^0$.

Set $\mathcal{T}_1^1 := \bigcup_j \mathcal{T}_j^0$ and observe that \mathcal{T}_1^1 is convex, $\{w_1, \dots, w_n\}$ -enveloped, and a minimal enveloping set for \mathcal{T}_1^1 contains strictly fewer vectors than a minimal enveloping set for \mathcal{T}_1^0 . If $\mathcal{T}_1^1 = \mathbb{Z}^2$, then the construction halts. If not, we continue inductively, defining sets

$$\mathcal{T}_1^1 \subset \mathcal{T}_2^1 \subset \dots$$

and edges $w_1^j \in E(\mathcal{T}_j^1)$ that contain at least N integer points and such that \mathcal{T}_{j+1}^1 is the w_1^j -extension of \mathcal{T}_j^1 . Let $\mathcal{T}_1^2 = \bigcup_j \mathcal{T}_j^1$. Again, any η -coloring of \mathcal{T}_1^0 extends uniquely to an η -coloring of \mathcal{T}_1^2 . Moreover \mathcal{T}_1^2 is $\{w_1, \dots, w_n\}$ -enveloped and has strictly fewer elements in a minimal enveloping set than \mathcal{T}_1^1 did. Thus we define convex set $\mathcal{T}_1^1 \subset \mathcal{T}_1^2 \subset \dots \subset \mathcal{T}_1^i$ with all of these properties. After at most n steps, $\mathcal{T}_1^i = \mathbb{Z}^2$, and any η -coloring of the finite set \mathcal{T}_1^1 extends uniquely to an η -coloring of \mathbb{Z}^2 .

Since the function $T^{\vec{u}}\eta$ is uniquely determined by $T^{\vec{u}}(\eta)|_{\mathcal{T}_1^1}$ and there are only finitely many η -colorings of \mathcal{T}_1^1 , the \mathbb{Z}^2 -orbit of η is finite. \square

3.3. Single periodicity. We have now developed the tools to prove Theorem 1.4. We recall the statement for convenience:

Theorem (Theorem 1.4). *Suppose $\eta \in \mathcal{A}^{\mathbb{Z}^2}$ and $X_\eta := \overline{\mathcal{O}(\eta)}$. If there exist $n, k \in \mathbb{N}$ such that $P_\eta(n, k) \leq nk$ and there is a unique nonexpansive 1-dimensional subspace for the \mathbb{Z}^2 -action (by translation) on X_η . Then η is periodic, but not doubly periodic,*

the unique nonexpansive line L is a rational line through the origin, and every period vector for η is contained in L .

We first define:

Definition 3.9. If $\mathcal{S} \subset \mathbb{Z}^2$ and $(a, b) \in \mathbb{Z}^2$, the (a, b) -diameter of \mathcal{S} is the number of distinct rational lines parallel to (a, b) that have nonempty intersection with \mathcal{S} . We denote this by $\text{diam}_{(a,b)}(\mathcal{S})$.

Proof. We adopt the same notation for \mathcal{S} , $w_1, \dots, w_n \in E(\mathcal{S})$, and $a, N \in \mathbb{N}$ used in the proof of Theorem 3.8.

We note that by Theorem 3.8, η is not doubly periodic. Moreover, by Corollary 3.4, the unique nonexpansive line ℓ is a rational line through the origin. Without loss of generality, we can assume that ℓ is vertical.

We claim that any η -coloring of $[1, a] \times [1, a]$ extends uniquely to an η -coloring of $[1, a] \times \mathbb{Z}$. Assuming the claim, the restriction of η to any vertical strip of width a is vertically periodic of period at most $P_\eta(\mathcal{T})$, where \mathcal{T} is the smallest $\{w_1, \dots, w_n\}$ -enveloped set containing $[1, a] \times [1, a]$, thereby completing the proof.

To prove the claim, let \mathcal{T}_1 be a $\{w_1, \dots, w_n\}$ -enveloped set that contains $[1, a] \times [1, a]$. Choose a non-vertical edge $w^1 \in E(\mathcal{T}_1)$ that contains at least N integer points (note that this is possible since $\text{diam}_{(0,1)}(\mathcal{T}_1) \geq a$). Let \mathcal{T}_2 be the w^1 extension of \mathcal{T}_1 . As in the proof of Theorem 3.8, any η -coloring of \mathcal{T}_1 extends uniquely to an η -coloring of \mathcal{T}_2 . Continuing inductively, we define $\{w_1, \dots, w_n\}$ -enveloped sets

$$\mathcal{T}_1 \subset \mathcal{T}_2 \subset \dots$$

and non-vertical edges $w^j \in E(\mathcal{T}_j)$ such that w^j contains at least N integer points and \mathcal{T}_{j+1} is the w^j -extension of \mathcal{T}_j . Then any η -coloring of the finite set \mathcal{T}_1 extends uniquely to an η -coloring of $\mathcal{T}_\infty := \bigcup \mathcal{T}_j$. Furthermore, \mathcal{T}_∞ is a convex, $\{w_1, \dots, w_n\}$ -enveloped subset of $[1, a] \times \mathbb{Z}$ which contains infinitely many integer points. Therefore \mathcal{T}_∞ is either the set $[1, a] \times \mathbb{Z}$ or contains the set $[1, a] \times [b, \infty)$ for some $b \in \mathbb{Z}$. In the former case, we are done. In the latter case, we proceed inductively, extending \mathcal{T}_∞ to $[1, a] \times \mathbb{Z}$. Then any η -coloring of \mathcal{T}_∞ extends uniquely to the set constructed at each stage, once again completing the proof. \square

Remark 3.10. We contrast this with a recent result of Hochman [11], which shows that there are \mathbb{Z}^2 -systems that have a unique nonexpansive 1-dimensional subspace but are not periodic. Theorem 1.4 only applies to the special case of those \mathbb{Z}^2 -subshifts that arise as the orbit closure of a function satisfying the hypothesis of Nivat's Conjecture.

4. A STRONGER BOUND ON COMPLEXITY

In light of Theorems 3.8 and 1.4, one strategy for proving Nivat's Conjecture is to show that if $\eta: \mathbb{Z}^2 \rightarrow \mathcal{A}$ satisfies $P_\eta(R_{n,k}) \leq nk$ for some $n, k \in \mathbb{N}$, then η does not have two linearly independent rigid directions. Under the strengthened hypothesis that $P_\eta(R_{n,k}) \leq \frac{nk}{2}$, this is the content of Theorem 1.2. The additional control over η obtained from the assumption $P_\eta(R_{n,k}) \leq \frac{nk}{2}$ comes in three guises: we obtain a special sort of η -generating set (Lemma 4.1), we prove the existence of sets that contain many points in any rational direction (Lemma 4.7 and Proposition 4.8), and we obtain control on the periods in Section 5.3.2.

4.1. Strong generating sets.

Lemma 4.1. *Suppose $\eta: \mathbb{Z}^2 \rightarrow \mathcal{A}$ is aperiodic and there exist $n, k \in \mathbb{N}$ such that $P_\eta(R_{n,k}) \leq \frac{nk}{2}$. Then there exists an η -generating set $\mathcal{S} \subset R_{n,k}$ such that*

- (i) $D_\eta(\mathcal{S}) \leq -\frac{|\mathcal{S}|}{2}$;
- (ii) For any $w \in E(\mathcal{S})$, the discrepancy function satisfies

$$D_\eta(\mathcal{S} \setminus \{w\}) \geq D_\eta(\mathcal{S}) + \left\lceil \frac{|w \cap \mathcal{S}|}{2} \right\rceil.$$

- (iii) If $\mathcal{T} \subset \mathcal{S}$ is convex and nonempty, then

$$D_\eta(\mathcal{T}) > D_\eta(\mathcal{S}).$$

We give a name to a set satisfying the conclusion of this lemma:

Definition 4.2. If $\eta: \mathbb{Z}^2 \rightarrow \mathcal{A}$ is aperiodic and satisfies $P_\eta(R_{n,k}) \leq \frac{nk}{2}$ for some $n, k \in \mathbb{N}$, then an η -generating set $\mathcal{S} \subseteq R_{n,k}$ is a *strong η -generating set* if it satisfies conditions (i), (ii), and (iii) of Lemma 4.1.

We note that the existence of such an η -generating is the first use of the stronger hypothesis on the complexity.

Proof of Lemma 4.1. We construct the set \mathcal{S} by an iterative process. By assumption we have $D_\eta(R_{n,k}) \leq -\frac{|R_{n,k}|}{2} = -\frac{nk}{2}$. Let $\mathcal{S}_1 \subseteq R_{n,k}$ be a convex set which is minimal (with respect to inclusion) among all convex subsets of $R_{n,k}$ that have discrepancy at most $D_\eta(R_{n,k})$. Minimality of \mathcal{S}_1 implies that \mathcal{S}_1 is η -generating. By construction, \mathcal{S}_1 satisfies $D_\eta(\mathcal{S}_1) \leq D_\eta(R_{n,k}) \leq -\frac{|R_{n,k}|}{2} \leq -\frac{|\mathcal{S}_1|}{2}$. If for every $w \in E(\mathcal{S}_1)$, the discrepancy satisfies $D_\eta(\mathcal{S}_1 \setminus \{w\}) \geq D_\eta(\mathcal{S}_1) + \left\lceil \frac{|w \cap \mathcal{S}_1|}{2} \right\rceil$, then the set $\mathcal{S} := \mathcal{S}_1$ satisfies the conclusions and we are finished.

Otherwise, suppose that we have inductively constructed a nested sequence of sets

$$R_{n,k} \supseteq \mathcal{S}_1 \supset \cdots \supset \mathcal{S}_m$$

such that for $i = 1, \dots, m$:

- (i) \mathcal{S}_i is convex and nonempty;
- (ii) \mathcal{S}_i is η -generating;
- (iii) \mathcal{S}_i satisfies $D_\eta(\mathcal{S}_i) \leq -\frac{|\mathcal{S}_i|}{2}$;
- (iv) \mathcal{S}_i is not the intersection of a line segment with \mathbb{Z}^2 ;
- (v) There exists $w_i \in E(\mathcal{S}_i)$ such that $D_\eta(\mathcal{S}_i \setminus \{w_i\}) < D_\eta(\mathcal{S}_i) + \left\lceil \frac{|w_i \cap \mathcal{S}_i|}{2} \right\rceil$.

Choose $w_m \in E(\mathcal{S}_m)$ such that $D_\eta(\mathcal{S}_m \setminus \{w_m\}) < D_\eta(\mathcal{S}_m) + \left\lceil \frac{|w_m \cap \mathcal{S}_m|}{2} \right\rceil$. Since the left hand side of the inequality is an integer,

$$D_\eta(\mathcal{S}_m \setminus \{w_m\}) < D_\eta(\mathcal{S}_m) + \frac{|w_m \cap \mathcal{S}_m|}{2}.$$

Let $\mathcal{S}_{m+1} \subset \mathcal{S}_m \setminus \{w_m\}$ be a convex set which is minimal (with respect to inclusion) among all convex subsets of $\mathcal{S}_m \setminus \{w_m\}$ of discrepancy at most $D_\eta(\mathcal{S}_m \setminus \{w_m\})$. Then $|\mathcal{S}_{m+1}| \leq |\mathcal{S}_m| - |w_m \cap \mathcal{S}_m|$, and so

$$\begin{aligned} D_\eta(\mathcal{S}_{m+1}) &\leq D_\eta(\mathcal{S}_m \setminus \{w_m\}) < D_\eta(\mathcal{S}_m) + \frac{|w_m \cap \mathcal{S}_m|}{2} \\ &\leq -\frac{|\mathcal{S}_m|}{2} + \frac{|w_m \cap \mathcal{S}_m|}{2} \leq -\frac{|\mathcal{S}_{m+1}|}{2}. \end{aligned}$$

By minimality, \mathcal{S}_{m+1} is η -generating, and contains at least two elements (since its η -discrepancy is negative). Thus we have satisfied conditions (i), (ii), and (iii). If \mathcal{S}_{m+1} is the intersection of a line segment with \mathbb{Z}^2 , then the Morse-Hedlund Theorem implies that the restriction of η to any line parallel to \mathcal{S}_{m+1} is periodic with period at most $|\mathcal{S}_{m+1}|$. But then η is periodic, a contradiction, and so condition (iv) is satisfied. If for every $w \in E(\mathcal{S}_{m+1})$,

$$D_\eta(\mathcal{S}_{m+1} \setminus \{w\}) \geq D_\eta(\mathcal{S}_{m+1}) + \left\lceil \frac{|w \cap \mathcal{S}|}{2} \right\rceil,$$

then the set $\mathcal{S} := \mathcal{S}_{m+1}$ satisfies the conclusions of the lemma. Otherwise \mathcal{S}_{m+1} satisfies all of the induction hypotheses and the construction continues. In both cases, condition (v) is satisfied.

Each \mathcal{S}_i is contained in $R_{n,k}$, so the construction terminates after finitely many steps. \square

Lemma 4.3. *Suppose $\eta: \mathbb{Z}^2 \rightarrow \mathcal{A}$ is aperiodic and there exist $n, k \in \mathbb{N}$ such that $P_\eta(R_{n,k}) \leq \frac{nk}{2}$. Let \mathcal{S} be a strong η -generating set. If $w \in E(\mathcal{S})$, then there are at most $\left\lceil \frac{|w \cap \mathcal{S}|}{2} \right\rceil$ distinct η -colorings of $\mathcal{S} \setminus \{w\}$ that extend non-uniquely to η -colorings of \mathcal{S} .*

Proof. The proof is identical to that of Lemma 2.7 with the stronger bound on $P_\eta(\mathcal{S}) - P_\eta(\mathcal{S} \setminus \{w\})$ implied by assumption that \mathcal{S} is strong generating. \square

Lemma 4.4. *Suppose $\eta: \mathbb{Z}^2 \rightarrow \mathcal{A}$, $\mathcal{S} \subset \mathbb{Z}^2$ is a strong η -generating set and there are antiparallel $w_1, w_2 \in E(\mathcal{S})$. Suppose $|w_1| \leq |w_2|$, H is a w_1 -half plane, and the restriction of $f \in X_\mathcal{S}(\eta)$ to H is (\mathcal{S}, η) -ambiguous. Then the $(\mathcal{S} \setminus \{w_1\}, w_1)$ -border of H is periodic with period vector parallel to w_1 . Its period is at most $\left\lceil \frac{|w \cap \mathcal{S}|}{2} \right\rceil$.*

Proof. Again, the proof is identical to that of Lemma 2.18 with the stronger bound on $P_\eta(\mathcal{S}) - P_\eta(\mathcal{S} \setminus \{w\})$ implied by the assumption on \mathcal{S} . \square

4.2. Balanced sets. We define a notion that captures a convex set that has maximal integer points on all lines parallel to its edges:

Definition 4.5. Suppose that $\eta: \mathbb{Z}^2 \rightarrow \mathcal{A}$ and $\mathcal{S} \subset \mathbb{Z}^2$ is finite and convex. Suppose ℓ is an oriented rational line and let $\ell(\mathcal{S}) \subseteq E(\mathcal{S}) \cup V(\mathcal{S})$ be the intersection of $\text{conv}(\mathcal{S})$ with the support line to \mathcal{S} parallel to ℓ . We say that \mathcal{S} is ℓ -balanced for η (or simply ℓ -balanced) if all of the following conditions hold:

- (i) Every rational line parallel to ℓ that has nonempty intersection with \mathcal{S} contains at least $|\ell(\mathcal{S}) \cap \mathcal{S}| - 1$ integer points;
- (ii) The endpoints of $\ell(\mathcal{S}) \cap \mathcal{S}$ are η -generated by \mathcal{S} ;
- (iii) $D_\eta(\mathcal{S} \setminus \{\ell(\mathcal{S})\}) > D_\eta(\mathcal{S})$.

Definition 4.6. For $\vec{v} \in \mathbb{Z}^2 \setminus \{\vec{0}\}$, a \vec{v} -strip is a convex subset of \mathbb{Z}^2 whose boundary contains precisely two edges, one of which is parallel to \vec{v} and the other is antiparallel to \vec{v} (we also include the degenerate case, calling the intersection of \mathbb{Z}^2 with a line parallel to \vec{v} , a \vec{v} -strip). The \vec{v} -width of a \vec{v} -strip is the number of distinct lines parallel to \vec{v} that have nonempty intersection with it (in the degenerate case, the width is 1).

Showing the existence of an ℓ -balanced set for η is the second use of the stronger hypothesis on complexity. It is used in the proof of Theorem 1.5 in Section 5.2.4.

Lemma 4.7. *If $\eta: \mathbb{Z}^2 \rightarrow \mathcal{A}$ and $P_\eta(R_{n,k}) \leq \frac{nk}{2}$ for some $n, k \in \mathbb{N}$, then for any rational line ℓ , there exists an ℓ -balanced set for η .*

Proof. If ℓ is a vertical line (without loss of generality, assume it points downward), choose minimal $k' \leq k$ such that $P_\eta(R_{n,k'}) \leq \frac{nk'}{2}$. Let $w \in E(R_{n,k'})$ be the edge parallel to ℓ . By minimality, $D_\eta(R_{n,k'} \setminus \ell) > D_\eta(R_{n,k'})$. Choose minimal \mathcal{S} satisfying

$$R_{n,k'} \setminus \ell \subset \mathcal{S} \subseteq R_{n,k'}$$

for which $D_\eta(\mathcal{S}) = D_\eta(R_{n,k'})$. By minimality of \mathcal{S} , the endpoints of the support line of \mathcal{S} parallel to ℓ are generated and \mathcal{S} satisfies the definition of an ℓ -balanced set. The case where ℓ is horizontal is similar.

If ℓ is neither vertical nor horizontal, assume, by composing η if needed with the map $(x, y) \mapsto (y, x)$, that $n \geq k$. Let $\vec{v} = (v_1, v_2) \in \mathbb{Z}^2$ be the shortest integer vector parallel to ℓ . Composing η with a map of the form $(x, y) \mapsto (\pm x, \pm y)$, if necessary, we assume that $v_1, v_2 < 0$. Assume that $v_2/v_1 > k/n$ (the other case is similar) and choose $\vec{u} \in \mathbb{Z}^2$ such that $\ell + \vec{u}$ passes through the

- northeast corner of $R_{n,k}$ if $v_2/v_1 > k/n$;
- southwest corner of $R_{n,k}$ if $v_2/v_1 \leq k/n$.

By choice of \vec{u} , $(\ell + \vec{u})$ intersects both the top and bottom of the rectangle $R_{n,k}$, so

$$\|\text{conv}(R_{n,k}) \cap (\ell + \vec{u})\| = \max_{\vec{v} \in \mathbb{Z}^2} \|\text{conv}(R_{n,k}) \cap (\ell + \vec{v})\|.$$

Moreover, one of the endpoints of the line segment $\text{conv}(R_{n,k}) \cap (\ell + \vec{u})$ is an integer point and so

$$(1) \quad |R_{n,k} \cap (\ell + \vec{u})| = \max_{\vec{v} \in \mathbb{Z}^2} |R_{n,k} \cap (\ell + \vec{v})|.$$

There is some $i \in \mathbb{N}$ such that $\ell + \vec{u} - (i, 0)$ passes through the southwest corner of $R_{n,k}$ and, by symmetry, the number of integer points in $R_{n,k}$ to the left of $\ell + \vec{u} - (i, 0)$ is the same as the number of integer points in $R_{n,k}$ to the right of $\ell + \vec{u}$ (see Figure 3).

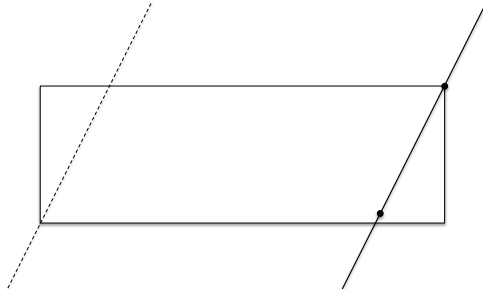


FIGURE 3. $R_{n,k}$ with $\ell + \vec{u}$ (solid line) and $\ell + \vec{u} - (i, 0)$ (dashed line) shown. The integer points in $R_{n,k}$ are preserved under the rotation by π about the center of $R_{n,k}$. The two points marked on $\ell + \vec{u}$ are the topmost and bottom most integer points of $(\ell + \vec{u}) \cap R_{n,k}$.

Let $\mathcal{S}_1 \subseteq R_{n,k}$ be the (convex) set of all $\vec{x} \in R_{n,k}$ that are either on $\ell + \vec{u}$ or to the left of it. Then $|R_{n,k} \setminus \mathcal{S}_1| \leq \frac{1}{2} |R_{n,k}|$ and so by Corollary 2.6, $D_\eta(\mathcal{S}_1) \leq 0$. Let $a, b \in \mathbb{Z}^2$ be the two extremal elements of $\mathcal{S}_1 \cap (\ell + \vec{u})$ (the dotted points in

Figure 3). Let $\mathcal{S}_2 \subseteq \mathcal{S}_1$ be minimal (with respect to inclusion) among all convex subsets of \mathcal{S}_1 that contain a and b and have η -discrepancy no larger than $D_\eta(\mathcal{S}_1)$. Then either $\mathcal{S}_2 = \mathcal{S}_1 \cap (\ell + \vec{u})$ or \mathcal{S}_2 contains $\mathcal{S}_1 \cap (\ell + \vec{u})$ and $\text{conv}(\mathcal{S}_2)$ has positive area. The case that $\text{conv}(\mathcal{S}_2)$ has positive area is illustrated in Figure 4.

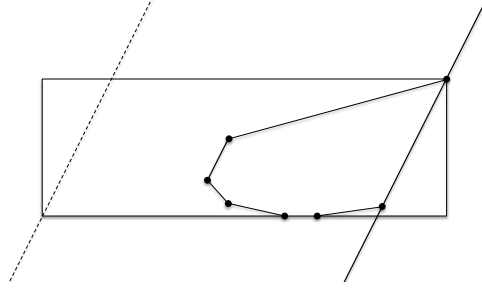


FIGURE 4. All points in \mathcal{S}_2 except the endpoints of $(\ell + \vec{u}) \cap R_{n,k}$ are η -generated, where ℓ points southwest and \mathcal{S}_2 is ℓ -balanced.

If the area of $\text{conv}(\mathcal{S}_2)$ is zero, let $\mathcal{S}_3 \subseteq \mathcal{S}_2$ be minimal among all convex subsets of \mathcal{S}_2 with η -discrepancy at most $D_\eta(\mathcal{S}_2)$. Then \mathcal{S}_3 is an η -generating set contained entirely in $\ell + \vec{u}$ and so \mathcal{S}_3 is ℓ -balanced.

In the second case, by minimality of \mathcal{S}_2 and Lemma 2.1, any vertex of \mathcal{S}_2 other than a and b must be η -generated by \mathcal{S}_2 . If $\ell(\mathcal{S}_2) \in V(\mathcal{S}_2)$, then $\ell(\mathcal{S}_2)$ is η -generated by \mathcal{S}_2 and so \mathcal{S}_2 is ℓ -balanced. Otherwise $\ell(\mathcal{S}_2) \in E(\mathcal{S}_2)$ and both of the extremal elements of $\ell(\mathcal{S}_2)$ are η -generated by \mathcal{S}_2 . Then $E(\mathcal{S}_2)$ has edges parallel and antiparallel to ℓ (the edge antiparallel to ℓ is the line segment $(\ell + \vec{u}) \cap \text{conv}(R_{n,k})$). By Equation (1), the number of integer points on the edge parallel to ℓ is no larger than the number of integer points on the edge antiparallel to ℓ . By Lemma 2.8, \mathcal{S}_2 is ℓ -balanced. \square

Balanced sets show that ambiguity gives rise to periodicity. In the following proposition, the assumption of the existence of an ℓ -balanced set \mathcal{S} is redundant (but convenient for the statement), as by applying Lemma 4.7, its existence follows directly from the assumption on complexity:

Proposition 4.8. *Suppose $\eta: \mathbb{Z}^2 \rightarrow \mathcal{A}$ and there exists $n, k \in \mathbb{N}$ such that $P_\eta(R_{n,k}) \leq \frac{nk}{2}$. Suppose ℓ is a rigid direction for η , \mathcal{S} is an ℓ -balanced set, and H is an ℓ -half plane. Then:*

- (i) *Any $f \in X_{\mathcal{S}}(\eta)$ whose restriction to H is (\mathcal{S}, η) -ambiguous is periodic with period vector parallel to ℓ .*
- (ii) *If $w \in E(\mathcal{S})$ is parallel to ℓ and $\tilde{\mathcal{S}} = \mathcal{S} \setminus \{w\}$, then the $(\tilde{\mathcal{S}}, w)$ -border of H has period at most $|w \cap \mathcal{S}| - 1$ and the restriction of f to any ℓ -strip of width $\text{diam}_w(\tilde{\mathcal{S}})$ has period at most $2|w \cap \mathcal{S}| - 2$.*

Proof. Without loss of generality, we can assume (see Remark 2.4) that w points vertically downward and $H = \{(x, y) \in \mathbb{Z}^2: x \geq 0\}$. Define $\tilde{\mathcal{S}} := \mathcal{S} \setminus \{w\}$ and for all $K \in \mathbb{Z}$, set

$$B_K := \left\{ (x, y) \in \mathbb{Z}^2: K \leq x < K + \text{diam}_w(\tilde{\mathcal{S}}) \right\}.$$

By translating if necessary, we can assume that $\tilde{\mathcal{S}} \subset B_0$. Let $h := |w \cap \mathcal{S}| - 1$.

We claim that $f|_{B_K}$ is periodic of period at most $2h$ for all $K \in \mathbb{Z}$, which establishes the lemma. We prove this using induction in several steps. We start by setting up the base case of the induction via two cases, depending on $f|_{B_K}$ extending uniquely or not.

4.2.1. *Assuming $f|_{B_K}$ does not extend uniquely.* If $f|_{B_K}$ does not extend uniquely to an (\mathcal{S}, η) -coloring of $B_K \cup B_{K-1}$, we claim that $f|_{B_K}$ is periodic with period at most h and $f|_{B_{K-1}}$ is periodic of period at most $2h$.

To prove this, suppose $K \in \mathbb{Z}$ and the coloring of B_K given by $f|_{B_K}$ does not extend uniquely to an (\mathcal{S}, η) -coloring of $B_K \cup B_{K+1}$. By Corollary 2.20, $f|_{B_K}$ is vertically periodic of period at most h . For the set \mathcal{S} , write $\mathcal{S}(i, j)$ for the translation $\mathcal{S} + (i, j)$, and we use the analogous notation for $\tilde{\mathcal{S}}$. Since \mathcal{S} is ℓ -balanced and the two endpoints of w are η -generated by \mathcal{S} , the coloring of $\tilde{\mathcal{S}}$ given by $f|_{\tilde{\mathcal{S}}}(-K, i)$ does not extend uniquely to an (\mathcal{S}, η) -coloring of \mathcal{S} for any $i \in \mathbb{Z}$ (otherwise we could use this information to deduce the coloring of B_{K-1}). This means that

$$\left| \left\{ f|_{\tilde{\mathcal{S}}}(0, j) : j \in \mathbb{Z} \right\} \right| \leq P_\eta(\mathcal{S}) - P_\eta(\tilde{\mathcal{S}}) \leq h.$$

Furthermore, the number of η -colorings of \mathcal{S} whose restriction to $\tilde{\mathcal{S}}$ is $(\tilde{\mathcal{S}}, \mathcal{S}, \eta)$ -ambiguous is at most $2h$, since each η -coloring of $\tilde{\mathcal{S}}$ extends to an η -coloring of \mathcal{S} , and the $P_\eta(\mathcal{S}) - P_\eta(\tilde{\mathcal{S}})$ extra colorings of \mathcal{S} all restrict to $(\tilde{\mathcal{S}}, \mathcal{S}, \eta)$ -ambiguous colorings of $\tilde{\mathcal{S}}$.

By the Pigeonhole Principle, there exist $0 \leq i < j \leq 2h - 1$ such that the η -colorings of \mathcal{S} given by $f|_{\mathcal{S}(K, i)}$ and $f|_{\mathcal{S}(K, j)}$ coincide. Recall that, since \mathcal{S} is ℓ -balanced, any vertical line $\tilde{\ell}$ that has nonempty intersection with \mathcal{S} satisfies $|\tilde{\ell} \cap \mathcal{S}| \geq h$ (Definition 4.5). Since the vertical period of $f|_{B_K}$ is at most h , $f|_{\tilde{\mathcal{S}}}(0, i+k)$ and $f|_{\tilde{\mathcal{S}}}(0, j+k)$ coincide for all k and since the endpoints of w are η -generated by \mathcal{S} , an easy induction argument shows that $f|_{\mathcal{S}(0, i+k)}$ and $f|_{\mathcal{S}(0, j+k)}$ coincide for all k . Thus $j - i \leq 2h$ is a period for $f|_{B_K \cup B_{K-1}}$ and the claim is proven.

4.2.2. *Assuming $f|_{B_K}$ extends uniquely.* If $f|_{B_K}$ is periodic and extends uniquely to an (\mathcal{S}, η) -coloring of $B_K \cup B_{K-1}$, we claim that $f|_{B_{K-1}}$ is periodic with period dividing that of $f|_{B_K}$.

To see this, suppose $K \in \mathbb{Z}$. We already know that

- (i) $f|_{B_K}$ is vertically periodic with period p ;
- (ii) The coloring of B_K given by $f|_{B_K}$ extends uniquely to an η -coloring of $B_K \cup \{(K-1, y) : y \in \mathbb{Z}\}$.

We claim that $f|_{B_{K-1}}$ is vertically periodic of period dividing p . If not, define $g : \mathbb{Z}^2 \rightarrow \mathcal{A}$ by

$$g(x, y) = \begin{cases} f(x, y) & \text{if } x \geq K; \\ f(x, y + p) & \text{if } x < K. \end{cases}$$

Since $f \in X_{\mathcal{S}}(\eta)$, $f|_{B_K}$ is periodic, and $\text{diam}_w(B_K) = \text{diam}_w(\tilde{\mathcal{S}})$, we are guaranteed that $g \in X_{\mathcal{S}}(\eta)$. Since the restriction of f to $\{(K-1, y) : y \in \mathbb{Z}\}$ is not periodic of period dividing p , $f|_{B_K} = g|_{B_K}$ but their restrictions to $B_K \cup \{(K-1, y) : y \in \mathbb{Z}\}$ do not agree, contradicting the fact that $f|_{B_K}$ had only one extension to an (\mathcal{S}, η) -coloring of B_{K-1} . This completes the proof of the claim.

4.2.3. Periodicity of $f|_{B_K}$ for $K < 0$. We now start the main induction, carried out in three steps. Starting with the ambiguity of $f|_H$, we show that the proposition holds for the restriction of f to $\mathbb{Z}^2 \setminus H$. We claim that $f|_{B_K}$ is vertically periodic of period at most $2h$ for all $K \leq 0$. Using induction to prove this claim, by Lemma 2.18, $f|_{B_0}$ is vertically periodic of period at most h . Suppose that $K < 0$ and for $i = 0, -1, \dots, K$, $f|_{B_i}$ is vertically periodic of period at most $2h$. One of the hypotheses of the two claims in step 0 applies to $f|_{B_K}$, and so $f|_{B_{K-1}}$ is periodic of period at most $2h$. By induction, this holds for all $K < 0$.

4.2.4. Periodicity of $f|_{B_K}$ for $K > 0$. To extend the result for $K > 0$, let \mathcal{S}_1 be a set balanced in the direction antiparallel to ℓ . Suppose that $\hat{w} \in E(\mathcal{S}_1)$ is antiparallel to ℓ and let $\tilde{\mathcal{S}}_1 := \mathcal{S}_1 \setminus \{\hat{w} \cap \mathcal{S}_1\}$. Define

$$\hat{B}_K := \{(x, y) \in \mathbb{Z}^2 : K - \text{diam}_{\hat{w}}(\mathcal{S}_1) + 1 \leq x \leq K\}$$

and assume, without loss of generality, that $\hat{w} \subset \{(0, y) : y \in \mathbb{Z}\}$. Then, by the result of 4.2.3, $f|_{\mathbb{Z}^2 \setminus H}$ is periodic and so $f|_{\hat{B}_0}$ is vertically periodic. By an induction argument analogous that given in Stages 0 and 1 (except now using \mathcal{S}_1 in place of \mathcal{S}), $f|_{\hat{B}_K}$ is periodic for all $K > 0$, and its period is at most the maximum of $(2|w \cap \mathcal{S}| - 2)!$ (an upper bound for the period of $f|_{\hat{B}_0}$) and $2|\hat{w} \cap \mathcal{S}_1| - 2$. This implies that there is some constant $C > 0$ such that for all $K \in \mathbb{Z}$, $f|_{B_K}$ is vertically periodic of period at most C . This establishes the first conclusion of the proposition.

4.2.5. Bounds on the period. To establish the second part of the proposition, we need an improvement on the bound of the vertical period of $f|_{B_K}$ when $K > 0$. For any $K_0 \in \mathbb{Z}$ such that $f|_{B_{K_0}}$ is vertically periodic of period at most $2h$, the induction argument from 4.2.3, but with the base case changed from B_0 to B_{K_0} , shows that $f|_{B_K}$ is vertically periodic of period at most $2h$, for all $K \leq K_0$. Therefore, it suffices to find a sequence $0 < i_1 < i_2 < \dots$ such that $f|_{B_{i_j}}$ is vertically periodic with period at most $2h$ for all $j \in \mathbb{N}$.

Assume instead that no such sequence exists. Then there exists $I \in \mathbb{N}$ such that for all $i > I$, the coloring of B_0 given by $(T^{(i,0)}f)|_{B_0}$ is either vertically aperiodic or periodic of period larger than $2h$. Since $f|_{B_0}$ is periodic of period at most $2h$, we have $I \geq 0$ and $(T^{(i,0)}f)|_{B_0}$ is vertically periodic of period at most $2h$. We can further assume that I is minimal with this property. For $i > I$, the fact that $f|_{B_i}$ does not satisfy the conclusion of Corollary 2.20 (specifically the bounds on its period) implies that $(T^{(i,0)}f)|_{B_0}$ extends uniquely to an (\mathcal{S}, η) -coloring of $B_0 \cup B_{-1}$. Since f is vertically periodic, there are only finitely many colorings of B_0 that occur as $(T^{(K,0)}f)|_{B_0}$, for $K \in \mathbb{Z}$. Thus there exists a smallest integer J such that $J \geq I$ and there is $j > J$ satisfying $(T^{(J,0)}f)|_{B_0} = (T^{(j,0)}f)|_{B_0}$. Since $(T^{(J,0)}f)|_{B_0}$ extends uniquely to an η -coloring of $B_0 \cup B_{-1}$, and since the functions $(T^{(J,0)}f)|_{B_0 \cup B_{-1}}$ and $(T^{(j,0)}f)|_{B_0 \cup B_{-1}}$ are two such colorings, they must coincide. Then $(T^{(J-1,0)}f)|_{B_0}$ coincides with $(T^{(j-1,0)}f)|_{B_0}$ and so $J = I$ (by minimality of J). Then $f|_{B_I} = f|_{B_J} = f|_{B_j}$ is periodic of period at most $2h$. But $f|_{B_j}$ is either aperiodic or periodic of period greater than $2h$, contradicting the definition of I . \square

Corollary 4.9. *Suppose $\eta: \mathbb{Z}^2 \rightarrow \mathcal{A}$ and there exists $n, k \in \mathbb{N}$ such that $P_\eta(R_{n,k}) \leq \frac{nk}{2}$. Suppose ℓ is a rational line, \mathcal{S} is an ℓ -balanced set, $w \in E(\mathcal{S})$ is parallel to ℓ ,*

and B is an ℓ -strip of width $\text{diam}_w(\mathcal{S}) - 1$. If $f \in X_{\mathcal{S}}(\eta)$ and $f|_B$ is periodic (with period vector parallel to ℓ), then f is periodic with period vector parallel to ℓ .

Proof. We proceed as in the proof of the first part of Proposition 4.8. The assumption of the corollary replaces the base case (4.2.1 and 4.2.2) and the induction steps of 4.2.3 and 4.2.4 are identical. \square

Lemma 4.10. *Suppose $\eta: \mathbb{Z}^2 \rightarrow \mathcal{A}$ and there exist $n, k \in \mathbb{N}$ such that $P_\eta(R_{n,k}) \leq \frac{nk}{2}$. If the oriented rational line ℓ is a rigid direction for η , then the direction antiparallel to ℓ is also rigid. In particular, any η -generating set has boundary edges parallel and antiparallel to ℓ .*

Proof. Let \mathcal{S} be an η -generating set, $w \in E(\mathcal{S})$ be parallel to ℓ , and without loss of generality, we can assume that ℓ points vertically downward. Let $\hat{\ell}$ be the direction antiparallel to ℓ and suppose for contradiction that $\hat{\ell}$ is not a rigid direction for η .

Set $H := \{(x, y) \in \mathbb{Z}^2: x \geq 0\}$. Since ℓ is a rigid direction, we can choose $f_1, f_2 \in X_{\mathcal{S}}(\eta)$ such that $f_1|_H = f_2|_H$ but $f_1 \neq f_2$. By Proposition 4.8, f_1 and f_2 are both vertically periodic. Since $f_1|_H = f_2|_H$, at most one of f_1 and f_2 is doubly periodic. Without loss of generality, assume that f_1 is not doubly periodic.

Since $\hat{\ell}$ is not rigid, there exist $a_1, a_2 \in \mathbb{N}$ and $A \in GL_2(\mathbb{Z})$ such that $\hat{\ell}$ is η -shiftable with parameters (a_1, a_2, A) . Then every η -coloring of a vertical strip of width at least $\|a_2 \cdot A^{-1}(1, 0)\|$ extends uniquely to an η -coloring of its $\hat{\ell}$ -extension. In particular, the restriction of f_1 to any vertical strip of this width extends uniquely. The vertical periodicity of f_1 implies that there are only finitely many such restrictions, each of which extends uniquely to its $\hat{\ell}$ -extension. So f_1 is also horizontally periodic, a contradiction. \square

Proposition 4.11. *Suppose $\eta: \mathbb{Z}^2 \rightarrow \mathcal{A}$ is aperiodic and there exist $n, k \in \mathbb{N}$ such that $P_\eta(R_{n,k}) \leq \frac{nk}{2}$. There exists a strong η -generating set \mathcal{S} such that if $w \in E(\mathcal{S})$ points in a rigid direction and ℓ is a rational line parallel to w having nonempty intersection with \mathcal{S} , then $|\ell \cap \mathcal{S}| \geq \left\lfloor \frac{|w \cap \mathcal{S}|}{2} \right\rfloor + 1$.*

Proof. Let \mathcal{S} be minimal (with respect to inclusion) among strong η -generating subset of $R_{n,k}$. By Lemma 4.10, if $w \in E(\mathcal{S})$ points in a rigid direction, then there exists $\hat{w} \in E(\mathcal{S})$ antiparallel to w . By convexity of \mathcal{S} , any rational line parallel to w that has nonempty intersection with \mathcal{S} satisfies

$$|\ell \cap \mathcal{S}| \geq \min \{|w \cap \mathcal{S}| - 1, |\hat{w} \cap \mathcal{S}| - 1\}.$$

If the right hand side of this inequality is at least two, then \mathcal{S} satisfies the conclusion of the proposition and we are done. Otherwise, one of $|w \cap \mathcal{S}|$ and $|\hat{w} \cap \mathcal{S}|$ is exactly two. Without loss of generality, assume that $|w \cap \mathcal{S}| = 2$. Suppose $w \cap \mathbb{Z}^2 = \{(x_1, y_1), (x_2, y_2)\}$. Then by Lemma 2.3,

$$D_\eta(\mathcal{S} \setminus \{(x_1, y_1)\}) = D_\eta(\mathcal{S}) + 1.$$

By rigidity of w , the vertex (x_2, y_2) is not η -generated by $\mathcal{S} \setminus \{(x_1, y_1)\}$. Thus $D_\eta(\mathcal{S} \setminus \{w\}) \leq D_\eta(\mathcal{S}) + 1$. On the other hand, $|\mathcal{S} \setminus \{w\}| = |\mathcal{S}| - 2$, and so $D_\eta(\mathcal{S} \setminus \{w\}) \leq -\left\lceil \frac{|\mathcal{S} \setminus \{w\}|}{2} \right\rceil$. But then $\mathcal{S} \setminus \{w\}$ contains a strong η -generating subset, a contradiction. \square

4.3. Constructions with balanced sets. We make precise what it means for a coloring to be periodic on a region:

Definition 4.12. Suppose that $\mathcal{T} \subset \mathbb{Z}^2$ is a convex set and there exists $\vec{v} \in \mathbb{Z}^2 \setminus \{\vec{0}\}$ such that $(\mathcal{T} + \vec{v}) \subseteq \mathcal{T}$. If $f: \mathcal{T} \rightarrow \mathcal{A}$ is an η -coloring of \mathcal{T} , then f is *periodic* of period $\vec{p} \in \mathbb{Z}^2 \setminus \{\vec{0}\}$ if $(\mathcal{T} + \vec{p}) \subseteq \mathcal{T}$ and $f(\vec{x}) = f(\vec{x} + \vec{p})$ for all $\vec{x} \in \mathcal{T}$.

If $w \in E(\mathcal{T})$, $\vec{u} \in \mathbb{Z}^2 \setminus \{\vec{0}\}$ is the shortest integer vector parallel to w , and $(\mathcal{T} + \vec{u}) \subseteq \mathcal{T}$, then $f|_{\mathcal{T}}$ is *w-eventually periodic with period $p \in \mathbb{N}$ and gap $g \in \mathbb{N}$* if $f|_{\mathcal{T} + g\vec{u}}$ is periodic with period $p\vec{u}$.

Definition 4.13. If $\mathcal{S} \subset \mathbb{Z}^2$ is a finite convex set and $w \in E(\mathcal{S})$, then a *semi-infinite (\mathcal{S}, w) -strip* is a set of the form

$$\vec{u} + \{\mathcal{S} + \lambda\vec{v}: \lambda \in \mathbb{N} \cup \{0\}\} \quad \text{or} \quad \vec{u} + \{\mathcal{S} - \lambda\vec{v}: \lambda \in \mathbb{N} \cup \{0\}\},$$

where $\vec{u} \in \mathbb{Z}^2$ and \vec{v} is the shortest integer vector parallel to w .

Pictorially, a semi-infinite (\mathcal{S}, w) -strip is a half-strip whose boundary edges are parallel to edges of \mathcal{S} (and not the other natural definition in which the boundary has two semi-infinite edges and one more edge connecting them).

Proposition 4.14. Suppose $\eta: \mathbb{Z}^2 \rightarrow \mathcal{A}$ and there exist $n, k \in \mathbb{N}$ such that $P_\eta(R_{n,k}) \leq \frac{nk}{2}$. Suppose ℓ is a rational line, \mathcal{S} is an ℓ -balanced set, and $w \in E(\mathcal{S})$ is parallel to ℓ . If \mathcal{T} is a semi-infinite $(\mathcal{S} \setminus \{w\}, w)$ -strip and $f \in X_{\mathcal{S}}(\eta)$ is such that $f|_{\mathcal{T}}$ does not extend uniquely to an η -coloring of the w -extension of \mathcal{T} , then $f|_{\mathcal{T}}$ is w -eventually periodic with gap at most $|w \cap \mathcal{S}| - 1$ and period at most $|w \cap \mathcal{S}| - 1$.

Moreover, if $\tilde{f} \in X_{\mathcal{S}}(\eta)$ and $\tilde{f}|_{\mathcal{T}}$ is eventually periodic of period at most $2|w \cap \mathcal{S}| - 2$, then any extension of $\tilde{f}|_{\mathcal{T}}$ to an η -coloring of the w -extension of \mathcal{T} is also w -eventually periodic with the same gap and period at most $2|w \cap \mathcal{S}| - 2$.

Proof. The first statement follows immediately from Corollary 2.20.

For the second, let $\tilde{\mathcal{S}} := \mathcal{S} \setminus \{w\}$. Without loss of generality, we can assume that w points vertically downward, the topmost element of w is $(0, 0)$, and $\tilde{f}|_{\mathcal{T}}$ is $(0, -1)$ -eventually periodic with period p and gap g . Suppose further that the boundary edge of \mathcal{T} parallel to w is $\{(0, y) \in \mathbb{Z}^2: y \leq 0\}$. (The case that the boundary edge of \mathcal{T} parallel to w is unbounded from above, rather than below, is analogous.)

Then $\tilde{f}|_{\mathcal{T} - (0, g)}$ is periodic with period $(0, -p)$. Let B be the $(\tilde{\mathcal{S}}, w)$ -border of $\mathcal{T} - (0, g)$. There exists $a \in \mathbb{Q}$ and $b \in \mathbb{N}$ such that the w -extension of \mathcal{T} is given by

$$\mathcal{T} \cup \{(x, y) \in \mathbb{Z}^2: -b \leq x, y \leq ax\}.$$

We proceed by induction. Let

$$B_i := \{(x, y) \in \mathbb{Z}^2: i \leq x < i + \text{diam}_w(\tilde{\mathcal{S}}), y \leq ax\}.$$

By assumption $\tilde{f}|_{B_0 - (0, g)}$ is vertically periodic with period at most $2|w \cap \mathcal{S}| - 2$. We claim that for all $i < 0$, if $\tilde{f}|_{B_i - (0, g)}$ is vertically periodic of period at most $2|w \cap \mathcal{S}| - 2$, then $\tilde{f}|_{B_{i-1} - (0, g)}$ is also . To prove the claim, we consider two cases.

4.3.1. *Unique extensions.* First we show that if $\tilde{f}|_{B_i - (0, g)}$ is periodic of period $p \leq 2|w \cap \mathcal{S}| - 2$ and there exists $j < -g$ such that the η -coloring of $\tilde{\mathcal{S}}$ given by $(T^{(i, j)}\tilde{f})|_{\tilde{\mathcal{S}}}$ extends uniquely to an η -coloring of \mathcal{S} , then $\tilde{f}|_{B_{i-1}}$ is periodic of period dividing p .

To see this, recall that the vertical period of $f|_{B_i - (0, g)}$ is p . Then $\tilde{f}|_{\mathcal{S} - (i, j + p)} = \tilde{f}|_{\mathcal{S} - (i, j)}$ and $\tilde{f}|_{\tilde{\mathcal{S}} - (i, j + l + p)} = \tilde{f}|_{\tilde{\mathcal{S}} - (i, j + l)}$ for all $l \in \mathbb{N}$ such that $l \leq ai - j$. Since \mathcal{S} is ℓ -balanced, the top most and bottom most elements of w are η -generated by \mathcal{S} . Thus $\tilde{f}|_{\mathcal{S} - (i, j + l)} = \tilde{f}|_{\mathcal{S} - (i, j)}$ for all such l . In particular, $\tilde{f}|_{(B_i \cup B_{i-1}) - (i, g)}$ is vertically periodic with period dividing p .

4.3.2. *No unique extensions.* Next we show that if there is no $j < -g$ for which the η -coloring of $\tilde{\mathcal{S}}$ given by $(T^{(i, j)}\tilde{f})|_{\tilde{\mathcal{S}}}$ extends uniquely to an η -coloring of \mathcal{S} , then $\tilde{f}|_{B_i - (0, g)}$ is periodic of period at most $|w \cap \mathcal{S}| - 1$ and $\tilde{f}|_{B_{i-1} - (0, g)}$ is periodic of period at most $2|w \cap \mathcal{S}| - 2$.

To prove this, if $\tilde{f}|_{B_i - (0, g)}$ does not extend uniquely to an η -coloring of the w -extension of $B_{i-1} - (0, g)$, then by Corollary 2.20, $\tilde{f}|_{B_i - (0, g)}$ is eventually periodic, and by our assumptions, we have that it is periodic with period at most $|w \cap \mathcal{S}| - 1$. As in the proof of Proposition 4.8, there are at most $2|w \cap \mathcal{S}| - 2$ distinct η -colorings of \mathcal{S} occurring as $\tilde{f}|_{\mathcal{S} - (i, y)}$ for $y \geq g$. By the Pigeonhole Principle, there exist $j, k \in \mathbb{N}$ with $g \leq j < k < g + 2|w \cap \mathcal{S}| - 2$ such that $\tilde{f}|_{\mathcal{S} - (i, j)} = \tilde{f}|_{\mathcal{S} - (i, k)}$. Since \mathcal{S} is ℓ -balanced, every vertical line that has nonempty intersection with \mathcal{S} intersects in at least $|w \cap \mathcal{S}| - 1$ places. Since $\tilde{f}|_{B_i - (0, g)}$ is periodic of period at most $|w \cap \mathcal{S}| - 1$, then $\tilde{f}|_{\tilde{\mathcal{S}} - (i, y)} = \tilde{f}|_{\tilde{\mathcal{S}} - (i, y + j - k)}$ for every $y \geq g$. Arguing as in the previous case, $\tilde{f}|_{B_i \cup B_{i-1} - (0, g)}$ is periodic of period at most $k - j \leq 2|w \cap \mathcal{S}| - 2$.

This proves the claim, and the result follows by induction. \square

Corollary 4.15. *Under the conditions of Proposition 4.14, if \mathcal{S} is an ℓ -balanced strong η -generating set, then $f|_{\mathcal{T}}$ is w -eventually periodic with gap at most $|w \cap \mathcal{S}| - 1$ and period at most $\left\lfloor \frac{|w \cap \mathcal{S}|}{2} \right\rfloor$.*

Proof. This is identical to the proof of Proposition 4.14, with the stronger bound on $P_\eta(\mathcal{S}) - P_\eta(\mathcal{S} \setminus \{w\})$ implied by the fact that \mathcal{S} is a strong η -generating set. \square

Corollary 4.16. *Suppose $\eta: \mathbb{Z}^2 \rightarrow \mathcal{A}$ and there exist $n, k \in \mathbb{N}$ such that $P_\eta(R_{n, k}) \leq \frac{nk}{2}$. Let ℓ be a rational line, \mathcal{S} an η -generating set, and $\vec{u} \in \mathbb{Z}^2$ be the shortest integer vector parallel to ℓ . Fix a finite set $F \subset \mathbb{Z}^2$ and an ℓ -strip B of width at least $\text{diam}_{\vec{u}}(\mathcal{S}) - 1$ that contains F . If there is some $f \in \overline{\mathcal{O}(\eta)}$ such that for all $\lambda \in \mathbb{Z}$ the coloring $(T^{\lambda \vec{u}} f)|_F$ extends uniquely to an η -coloring of B , then η is periodic.*

Proof. The condition on f guarantees that $f|_B$ is periodic with period vector parallel to \vec{u} . Since $f \in \overline{\mathcal{O}(\eta)}$, there is a translation $\vec{v} \in \mathbb{Z}^2$ such that $(T^{\vec{v}} \eta)|_F = f|_F$ and by uniqueness $(T^{\vec{v}} \eta)|_B = f|_B$. By Corollary 4.9, $T^{\vec{v}} \eta$ is periodic with period vector parallel to \vec{u} . Therefore η is also. \square

5. PROOF OF THEOREM 1.5

5.1. Starting the proof of Theorem 1.5. The proof of Theorem 1.5 is completed in this section via multiple steps; we include a short summary of what is covered at the beginning of each section. We proceed by contradiction, and the rough overall structure of the proof is as follows: assuming the existence of a counterexample, we produce other counterexamples with more structure (specifically with large regions on which they are periodic). With a sufficiently well structured counterexample, we fix a generating set and count colorings of it that occur on the boundary of the region of periodicity. We reach a contradiction by showing that a larger than possible number of distinct colorings occur.

Throughout this section, we assume that η is a counterexample to Theorem 1.5, meaning that $\eta : \mathbb{Z}^2 \rightarrow \mathcal{A}$ has at least two linearly independent rigid directions and there exist $n, k \in \mathbb{N}$ such that $P_\eta(R_{n,k}) \leq \frac{nk}{2}$. We remark that if η were periodic, it could have at most one rigid direction. Therefore we can assume that η is aperiodic.

5.2. An aperiodic counterexample with doubly periodic regions. We use the existence of η to construct $\alpha \in \overline{\mathcal{O}(\eta)}$ which is aperiodic, but the restriction of α to a large convex subset of \mathbb{Z}^2 is doubly periodic. This is carried out in three steps, first showing the existence of $f \in \overline{\mathcal{O}(\eta)}$ which is singly, but not doubly, periodic (Section 5.2.1) and then using f to show that there exists an aperiodic $\alpha \in \overline{\mathcal{O}(\eta)}$ that is doubly periodic on a large convex region (Sections 5.2.2 and 5.2.3).

5.2.1. A periodic half plane. By Lemma 4.1, there exists a strong η -generating set S . Let ℓ_1 be a rigid direction for η . By Lemma 3.3, there is some $w_1 \in E(\mathcal{S})$ parallel to ℓ_1 . By Lemma 4.10, the direction antiparallel to ℓ_1 is also rigid and there is some $w_2 \in E(\mathcal{S})$ antiparallel to ℓ_1 . By convexity, \mathcal{S} is either w_1 -balanced or w_2 -balanced. Without loss of generality,

(2) we assume that \mathcal{S} is w_1 -balanced

and that (see Remark 2.4) w_1 points vertically downward.

Set $\tilde{\mathcal{S}} := \mathcal{S} \setminus \{w_1\}$ and set

$$\begin{aligned} H_i &:= \{(x, y) \in \mathbb{Z}^2 : x \geq i\}; \\ A_i &:= \{(x, y) \in \mathbb{Z}^2 : i \leq x < i + \text{diam}_{w_1}(\tilde{\mathcal{S}})\}. \end{aligned}$$

By Lemma 3.2, there exist $f, g \in \overline{\mathcal{O}(\eta)}$ such that $f|_{H_1} = g|_{H_1}$ but $f|_{H_0} \neq g|_{H_0}$. At most one of $f|_{H_0}$ and $g|_{H_0}$ has a horizontal period vector (in the sense of Definition 4.12). Thus we can assume that $f|_{H_0}$ is not horizontally periodic. By Proposition 4.8, f is vertically periodic, and for every $i \in \mathbb{Z}$, $f|_{A_i}$ has period at most $2|w_1 \cap \mathcal{S}| - 2$. Moreover, by Lemma 4.4, the vertical period of $f|_{A_1}$ is at most $\left\lfloor \frac{|w_1 \cap \mathcal{S}|}{2} \right\rfloor$. By Proposition 4.11, we can assume that $\left\lfloor \frac{|w_1 \cap \mathcal{S}|}{2} \right\rfloor \leq |w_1 \cap \mathcal{S}| - 2$.

We also remark that, by the bound on the vertical period of $f|_{A_i}$ for $i \geq 0$, if $G \subset H_0$ is a convex set such that

(H-I) $(G + (1, 0)) \subset G$;

(H-II) G contains at least $2|w_1 \cap \mathcal{S}| - 2$ points on the y -axis,

then $f|_G$ does not have a horizontal period vector in the sense of Definition 4.12.

We summarize the main features of this construction:

- (i) $f \in \overline{\mathcal{O}(\eta)}$;
- (ii) f is vertically periodic;

- (iii) $f|_{A_1}$ is vertically periodic of period at most $|w_1 \cap \mathcal{S}| - 2$;
- (iv) The restriction of f to an infinite convex set $G \subset H_0$ that satisfies conditions (H-I) and (H-II) cannot be extended to a horizontally periodic η -coloring of \mathbb{Z}^2 .

5.2.2. *Construction of α .* Translating \mathcal{S} if necessary, we may assume that $(0, 0) \in w_1 \subset \{(0, y) : y \in \mathbb{Z}\}$. Using an inductive procedure, we define a function $\alpha \in \overline{\mathcal{O}(\eta)}$ which is aperiodic but agrees with f on an infinite, convex subset of \mathbb{Z}^2 (and is, therefore, vertically periodic on this subset).

Base case: Let $F_1 := \mathcal{S}$ and $G_0 = (0, 0)$. By Corollary 4.16 and aperiodicity of η , there exists $y_1 \in \mathbb{Z}$ such that $(T^{(0, y_1)} f)|_{F_1}$ does not extend uniquely to an η -coloring of the strip

$$B_1 := \{(x, y) \in \mathbb{Z}^2 : 0 \leq x < \text{diam}_{w_1}(\mathcal{S})\}.$$

Let $\alpha_1 \in \overline{\mathcal{O}(\eta)}$ be such that $\alpha_1|_{F_1} = (T^{(0, y_1)} f)|_{F_1}$, but $\alpha_1|_{B_1} \neq (T^{(0, y_1)} f)|_{B_1}$. Let G_1 be a maximal, $E(\mathcal{S})$ -enveloped subset of B_1 that contains F_1 and is such that $\alpha_1|_{G_1} = (T^{(0, y_1)} f)|_{G_1}$.

Inductive step: Suppose that we have constructed sequences of convex, $E(\mathcal{S})$ -enveloped, finite sets

$$G_0 \subseteq F_1 \subseteq G_1 \subseteq F_2 \subseteq G_2 \subseteq \cdots \subseteq F_i \subseteq G_i,$$

functions $\alpha_1, \dots, \alpha_i \in \overline{\mathcal{O}(\eta)}$, and integers y_1, \dots, y_i such that for $1 \leq j \leq i$:

- (i) (F_j hypothesis) F_j contains both G_{j-1} and $[0, j-1] \times [-j+1, j-1]$;
- (ii) (α_j hypothesis) Defining the strip B_j by

$$B_j := \{(x, y) \in \mathbb{Z}^2 : 0 \leq x < \text{diam}_{w_1}(F_j)\},$$

then

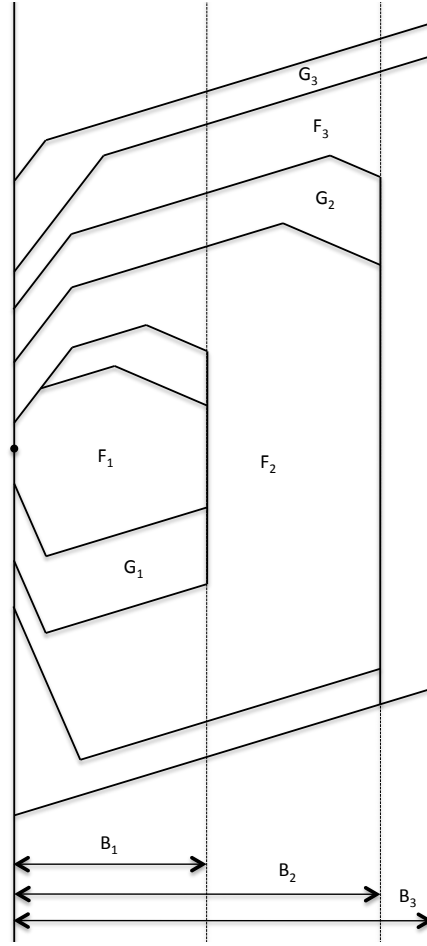
- (a) $\alpha_j|_{F_j} = (T^{(0, y_j)} f)|_{F_j}$;
- (b) $\alpha_j|_{B_j} \neq (T^{(0, y_j)} f)|_{B_j}$;
- (iii) (G_j hypothesis) $G_j \subset B_j$ is a maximal set among all convex, $E(\mathcal{S})$ -enveloped subsets of B_j such that
 - (a) $F_j \subseteq G_j$;
 - (b) $\alpha_j|_{G_j} = (T^{(0, y_j)} f)|_{G_j}$.

Let $F_{j+1} \subset H_0$ be a finite, convex, $E(\mathcal{S})$ -enveloped set containing both G_j and $[0, j] \times [-j, j]$. By Corollary 4.16 and aperiodicity of η , there exists $y_{j+1} \in \mathbb{Z}$ such that $(T^{(0, y_{j+1})} f)|_{F_{j+1}}$ does not extend uniquely to an η -coloring of the strip

$$B_{j+1} := \{(x, y) \in \mathbb{Z}^2 : 0 \leq x < \text{diam}_{w_1}(F_{j+1})\}.$$

Choose $\alpha_{j+1} \in \overline{\mathcal{O}(\eta)}$ such that $\alpha_{j+1}|_{F_{j+1}} = (T^{(0, y_{j+1})} f)|_{F_{j+1}}$, but $\alpha_{j+1}|_{B_{j+1}} \neq (T^{(0, y_{j+1})} f)|_{B_{j+1}}$. Let $G_{j+1} \subset B_{j+1}$ be a maximal $E(\mathcal{S})$ -enveloped set containing F_{j+1} such that $\alpha_{j+1}|_{G_{j+1}} = (T^{(0, y_{j+1})} f)|_{G_{j+1}}$. By induction these functions, sets, and integers are defined for all j .

By vertical periodicity of f , we can assume that $y_j \in [0, (2|w_1 \cap \mathcal{S}| - 2)!) for all $j \in \mathbb{Z}$. By passing to a subsequence, we can assume that the sequence $\{y_j\}_{j \in \mathbb{N}}$ is constant and, by replacing f with $T^{(0, y_1)} f$ if necessary, we can assume that this constant is zero.$

FIGURE 5. The sets $G_0 \subseteq F_1 \subseteq G_1 \subseteq \dots \subseteq G_3$.

By construction, for each $j \in \mathbb{N}$, $E(G_j)$ has a downward oriented edge contained in the y -axis. Let $(0, z_j) \in \mathbb{Z}^2$ be the topmost element of this edge and let

$$\tilde{G}_j := \{(x, y - z_j) : (x, y) \in G_j\}.$$

Then $(T^{(0, z_j)} \alpha_j) \upharpoonright \tilde{G}_j = (T^{(0, z_j)} f) \upharpoonright \tilde{G}_j$ and there is no $E(\mathcal{S})$ -enveloped convex subset of B_j that strictly contains \tilde{G}_j for which this is true (by maximality of G_j). By vertical periodicity of f , $\{T^{(0, z_j)} f : j \in \mathbb{N}\} \subseteq \{T^{(0, m)} f : 0 \leq m < (2|w_1 \cap \mathcal{S}| - 2)!\}$. Let $z \in [0, (2|w_1 \cap \mathcal{S}| - 2)!) be such that $T^{(0, z_j)} f = T^{(0, z)} f$ for infinitely many j . By passing to a subsequence, we can assume this holds for all j . Define $\tilde{\alpha}_j := T^{(0, z_j)} \alpha_j$ and $\tilde{f} := T^{(0, z)} f$. Then with this notation, $\tilde{\alpha}_j \upharpoonright \tilde{G}_j = \tilde{f} \upharpoonright \tilde{G}_j$ and there is no $E(\mathcal{S})$ -enveloped subset of B_j that strictly contains \tilde{G}_j for which this holds.$

Enumerate the vectors in $E(\mathcal{S})$ as u_1, u_2, \dots, u_m where $u_1 = w_1$ and $u_{k+1} = \text{pred}(u_k)$ for $k = 1, \dots, m-1$ (recall that $\partial\mathcal{S}$ is positively oriented and $\text{pred}(\cdot)$ is the predecessor edge with this orientation). Let $K \in \mathbb{N}$ be the index for which $u_K = w_2$ (the edge of $\partial\mathcal{S}$ antiparallel to w_1). Since \tilde{G}_j is $E(\mathcal{S})$ -enveloped for all j ,

define

$$h(j, k) = \begin{cases} \|a_{j,k}\| & \text{if there is some } a_{j,k} \in E(\tilde{G}_j) \text{ parallel to } u_k; \\ 0 & \text{otherwise.} \end{cases}$$

Passing to a subsequence if necessary, we can assume that for each fixed $k = 1, 2, \dots, m$, the function $h(\cdot, k)$ is either constant or strictly increasing as a function of j . By construction, $[0, j] \times [-j, j] \subseteq G_{j+1} \subset H_0$ for all j . So $\bigcup_j G_j = H_0$ and there is at least one index $1 < k < K$ for which $h(\cdot, k)$ is unbounded.

(3) Let $1 < k_{\min} < K$ be the least integer for which this holds.

Define integers $1 < k_1 < \dots < k_s < k_{\min}$ to be the indices in this interval for which $h(\cdot, k)$ is eventually positive. Let $v_1, \dots, v_s \in E(G_1)$ be the edges for which v_i is parallel to u_{k_i} . We emphasize that by construction, $v_1, \dots, v_s \in E(G_i)$ for all $i \geq 1$, meaning that not only does G_i have an edge parallel to v_1, v_2, \dots , but these fixed line segments are edges of G_i . Set

$$G_\omega := \bigcup_{j=1}^{\infty} \tilde{G}_j.$$

Then G_ω is convex and $E(\mathcal{S})$ -enveloped (see Figure 6). Moreover $E(G_\omega)$ is comprised of v_1, \dots, v_s , as well as $\{(0, y) \in \mathbb{Z}^2 : y \leq 0\}$, and a semi-infinite edge parallel to $u_{k_{\min}}$.

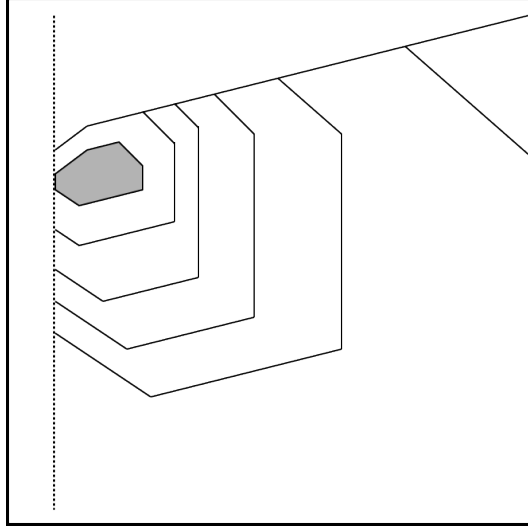


FIGURE 6. The set \mathcal{S} is shaded, and the sets $\mathcal{S} \subseteq \tilde{G}_1 \subseteq \tilde{G}_2 \subseteq \dots \subset G_\omega$ are shown.

By compactness, the sequence $\{\tilde{\alpha}_j\}_{j \in \mathbb{N}}$ has an accumulation point. Let $\alpha \in \overline{\mathcal{O}(\eta)}$ be such a point. By passing to a subsequence, we can assume that for all $1 \leq j_1 < j_2$ we have $\tilde{\alpha}_{j_2} \upharpoonright \tilde{G}_{j_1} = \tilde{\alpha}_{j_1} \upharpoonright \tilde{G}_{j_1}$. By construction, $\alpha \upharpoonright G_\omega = \tilde{f} \upharpoonright G_\omega$. In particular, $\alpha \upharpoonright G_\omega$ is vertically periodic (in the sense of Definition 4.12) and the restriction of α to any semi-infinite $(\tilde{\mathcal{S}}, w_1)$ -strip in G_ω has period at most $2|w_1 \cap \mathcal{S}| - 2$. Moreover, the restriction of α to the $(\tilde{\mathcal{S}}, w_1)$ -border of G_ω has period at most $|w_1 \cap \mathcal{S}| - 2$.

(because $\tilde{f} = T^{(0,z)}f$, the $(\tilde{\mathcal{S}}, w_1)$ -border of G_ω is a subset of A_1 , and this bound on the period was shown for $f|_{A_1}$ in Section 5.2.1).

5.2.3. *A second ambiguous direction for α .* Next, we show that both semi-infinite edges in $E(G_\omega)$ are η -rigid.

Let $\text{Ext}_{u_{k_{\min}}}(G_\omega)$ denote the $u_{k_{\min}}$ -extension of G_ω (recall Definition 2.15). Since the boundary edge of $\text{Ext}_{u_{k_{\min}}}(G_\omega)$ parallel to $u_{k_{\min}}$ is semi-infinite, we have that $\text{Ext}_{u_{k_{\min}}}(G_\omega) \setminus G_\omega$ is equal to the intersection of \mathbb{Z}^2 with the disjoint union of finitely many semi-infinite lines l_1, \dots, l_{r_1} parallel to $u_{k_{\min}}$. We denote the subextensions by

$$(4) \quad G_\omega^{(i)} := G_\omega \cup (l_1 \cap \mathbb{Z}^2) \cup \dots \cup (l_i \cap \mathbb{Z}^2)$$

for $i = 1, \dots, r_1$.

We now inductively define an increasing sequence of sets $\{G_\omega^{(i)}\}_{i \in \mathbb{N}}$. Suppose we have constructed integers $r_1, \dots, r_m \in \mathbb{N}$ and an increasing sequence of convex sets $\{G_\omega^{(i)}\}_{i=1}^{r_1+\dots+r_m}$ such that for all $j = 1, \dots, m$, the sets $G_\omega^{(r_1+\dots+r_j)}$ are $E(\mathcal{S})$ -enveloped and each has a semi-infinite edge parallel to $u_{k_{\min}}$ (the edge is not required to be the same for all of the sets). Then $\text{Ext}_{u_{k_{\min}}}(G_\omega^{(r_1+\dots+r_m)}) \setminus G_\omega^{(r_1+\dots+r_m)}$ is nonempty and can be written as the intersection of \mathbb{Z}^2 with the disjoint union of r_{m+1} semi-infinite lines $l_{r_1+\dots+r_m+1}, \dots, l_{r_1+\dots+r_m+1+r_{m+1}}$ (for some $r_{m+1} \in \mathbb{N}$). For $r_1 + \dots + r_m < i \leq r_1 + \dots + r_{m+1}$ define

$$G_\omega^{(i)} := G_\omega \cup (l_1 \cap \mathbb{Z}^2) \cup \dots \cup (l_i \cap \mathbb{Z}^2).$$

This defines a sequence of integers $\{r_m\}_{m \in \mathbb{N}}$ and sets $\{G_\omega^{(i)}\}_{i \in \mathbb{N}}$.

Recall that for all j , \tilde{G}_j is $E(\mathcal{S})$ -enveloped and the length of the boundary edge parallel to $u_{k_{\min}}$ increases monotonically in j (by (3)). Thus for j sufficiently large, $\text{Ext}_{u_{k_{\min}}}(\tilde{G}_j) \neq \tilde{G}_j$. Moreover the depth of the extension $\text{Ext}_{u_{k_{\min}}}(\tilde{G}_j)$ depends only on the slopes of the lines determined by the boundary edges of \tilde{G}_j (recall Definition 2.15). Therefore if d_j is the depth of the extension $\text{Ext}_{u_{k_{\min}}}(\tilde{G}_j)$, then d_j is bounded (in j) and there is some $d \in \mathbb{N}$ such that $d_j = d$ for infinitely many j . We pass to a subsequence such that this holds for all j .

Let $\tilde{l}(j, k)$ be the intersection of $\text{Ext}_{u_{k_{\min}}}(\tilde{G}_j)$ and l_k . Then $\text{Ext}_{u_{k_{\min}}}(G_j)$ can be written as the disjoint union

$$\tilde{G}_j \sqcup \bigsqcup_{k=1}^d \tilde{l}(j, k).$$

By construction (recall the inductive hypotheses for G_j at the beginning of this subsection) $f|_{\text{Ext}_{u_{k_{\min}}}(\tilde{G}_j)} \neq \alpha|_{\text{Ext}_{u_{k_{\min}}}(\tilde{G}_j)}$. Let $1 \leq a \leq d$ be the smallest integer for which $\tilde{\alpha}_j|_{\tilde{l}(j, a)} \neq \tilde{f}|_{\tilde{l}(j, a)}$ for infinitely many j . Passing to a subsequence, we can assume that for all $j \in \mathbb{N}$, $\tilde{\alpha}_j|_{\tilde{l}(j, k)} = \tilde{f}|_{\tilde{l}(j, k)}$ for all $1 \leq k < a$, but $\tilde{\alpha}_j|_{\tilde{l}(j, a)} \neq \tilde{f}|_{\tilde{l}(j, a)}$. Let

$$(5) \quad w_3 \in E(\mathcal{S}) \text{ be the edge parallel to } u_{k_{\min}}.$$

By Corollary 3.7, there are never $|w_3 \cap \mathcal{S}| - 1$ consecutive integer points on $\tilde{l}(j, a)$ where $\tilde{\alpha}_j$ and \tilde{f} coincide (otherwise they would coincide everywhere on $\tilde{l}(j, a)$ since \mathcal{S} is η -generating). In particular, there are never $|w_3 \cap \mathcal{S}| - 1$ consecutive integer

points on l_a where α and \tilde{f} coincide. As a result, the restriction of $\alpha|_{G_\omega^{(a-1)}}$ does not extend uniquely to an η -coloring of $G_\omega^{(a)}$.

Moreover, since f , and hence \tilde{f} , is vertically periodic and $\alpha|_{G_\omega^{(a-1)}} = \tilde{f}|_{G_\omega^{(a-1)}}$ but $\alpha|_{G_\omega^{(a)}} \neq \tilde{f}|_{G_\omega^{(a)}}$, α is not vertically periodic. Moreover, because the boundary edge of $G_\omega^{(a-1)}$ parallel to w_3 is semi-infinite there is an ambiguous coloring of a w_3 -half plane, obtained by passing to appropriate accumulation points of $\{T^{-m \cdot w_3} \alpha\}_{m=1}^\infty$ and $\{T^{-m \cdot w_3} \tilde{f}\}_{m=1}^\infty$ (viewing w_3 as a vector). Thus by Lemma 3.2, w_3 is a rigid direction for η and by Lemma 4.10, there is an edge $w_4 \in E(\mathcal{S})$ antiparallel to w_3 .

5.2.4. Construction of K . We show that there is an infinite, convex subset K of G_ω such that $\alpha|_K$ is doubly periodic. The construction has four steps which are illustrated in Figure 7.

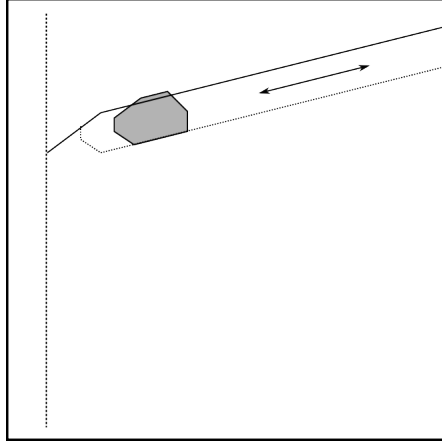
Step 1: By Lemma 4.7, there exists a w_3 -balanced set \mathcal{S}_1 . Let $\hat{w}_3 \in E(\mathcal{S}_1)$ be the edge parallel to w_3 and let $\tilde{\mathcal{S}}_1 := \mathcal{S}_1 \setminus \{\hat{w}_3\}$. Recall the integer $a \in \mathbb{N}$ defined in Section 5.2.3 is such that $\tilde{f}|_{G_\omega^{(a-1)}} = \alpha|_{G_\omega^{(a-1)}}$ but $\tilde{f}|_{G_\omega^{(a)}} \neq \alpha|_{G_\omega^{(a)}}$. By Lemma 4.14, the $(\tilde{\mathcal{S}}_1, \hat{w}_3)$ -border of $G_\omega^{(a-1)}$ is w_3 -eventually periodic. This is illustrated in Figure 7A.

Step 2: Let \mathcal{B} denote the $(\tilde{\mathcal{S}}_1, \hat{w}_3)$ -border of $G_\omega^{(a-1)}$. Since \tilde{f} is vertically periodic and $\tilde{f}|_{G_\omega^{(a-1)}} = \alpha|_{G_\omega^{(a-1)}}$, $\alpha|_{G_\omega^{(a-1)}}$ is vertically periodic (in the sense of Definition 4.12). Let $p \in \mathbb{N}$ be the minimal vertical period of $\alpha|_{G_\omega^{(a-1)}}$ such that $(T^{(0,p)} \alpha)|_{(T^{(0,-p)} G_\omega^{(a-1)})} = \alpha|_{(T^{(0,-p)} G_\omega^{(a-1)})}$. Then the restriction of α to any set of the form $T^{(0,-mp)} \mathcal{B}$ is eventually w_3 -periodic, with the same eventual period and the same gap. This is illustrated in Figure 7B.

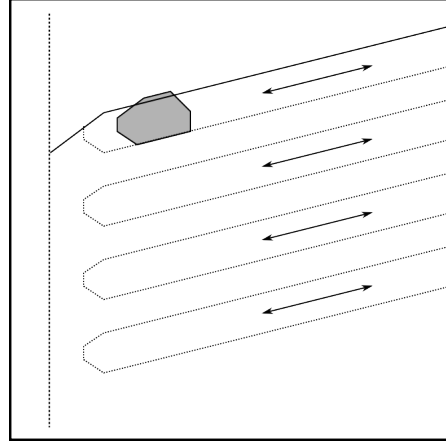
Step 3: The set $T^{(0,-p)} \mathcal{B}$ is a semi-infinite $(\tilde{\mathcal{S}}_1, \hat{w}_3)$ -strip (recall Definition 4.13). Let $\mathcal{B}_1 := \text{Ext}_{w_3}(T^{(0,-p)} \mathcal{B})$ be the w_3 -extension of \mathcal{B} . Now inductively let $\mathcal{B}_{i+1} := \text{Ext}_{w_3}(\mathcal{B}_i)$ for $i \in \mathbb{N}$. Then there is some $j \in \mathbb{N}$ such that \mathcal{B}_j contains all but finitely many elements of $G_\omega^{(a-1)} \setminus T^{(0,-p)} G_\omega^{(a-1)}$. Since $\alpha|_{T^{(0,-p)} \mathcal{B}}$ is eventually w_3 -periodic, Proposition 4.14 guarantees that $\alpha|_{\mathcal{B}_j}$ is also eventually w_3 -periodic with the same gap but possibly larger eventual period. This is illustrated in Figure 7C.

Step 4: Since $\alpha|_{G_\omega^{(a-1)}}$ is vertically periodic with period p , the restriction of α to $\bigcup_{m=1}^\infty T^{(0,-mp)}(\mathcal{B}_{j+1} \cap G_\omega^{(a-1)})$ is eventually w_3 -periodic. Hence there is some $q \in \mathbb{N}$ such that the restriction of α to $T^{(q,0)} \bigcup_{m=1}^\infty T^{(0,-mp)}(\mathcal{B}_{j+1} \cap G_\omega^{(a-1)})$ is w_3 -periodic. Since \mathcal{B}_{j+1} is a semi-infinite w_3 -strip and $\mathcal{B}_{j+1} \cap T^{(0,-p)} \mathcal{B}_{j+1} \neq \emptyset$, there is an $E(\mathcal{S})$ -enveloped, convex set $\tilde{K} \subseteq T^{(q,0)} \bigcup_{m=1}^\infty T^{(0,-mp)}(\mathcal{B}_{j+1} \cap G_\omega^{(a-1)})$ whose boundary has semi-infinite edges parallel to w_3 and w_1 . The restriction of α to \tilde{K} is doubly periodic. Let K be the largest (with respect to inclusion) convex set containing \tilde{K} for which $\alpha|_K$ is doubly periodic. By construction $\alpha|_{G_\omega^{(a)}}$ is not doubly periodic since it differs from the vertically periodic coloring \tilde{f} . Therefore K has a semi-infinite edge parallel to w_3 . Since $\alpha|_{G_\omega^{(a-1)}} = \tilde{f}|_{G_\omega^{(a-1)}}$ was constructed such that it is not horizontally periodic (in the sense of Definition 4.12), the set K has a semi-infinite edge parallel to w_1 . This is illustrated in Figure 7D.

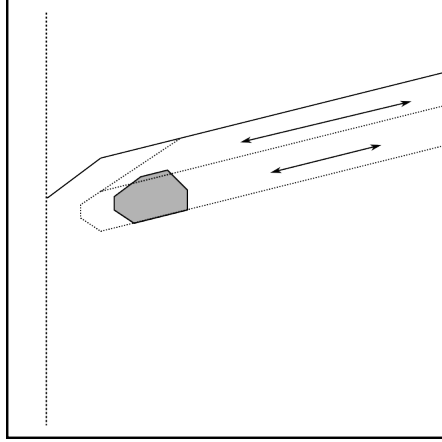
5.3. Bounds on the period of α . In this section, we show that we have strong bounds on the w_1 - and w_3 -periods of $\alpha|_K$, first extending the region K to a larger



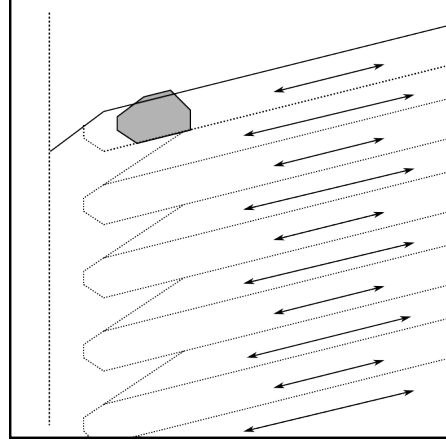
(A) The shaded set \mathcal{S}_1 and the set $G_\omega^{(a-1)}$ with its $(\tilde{\mathcal{S}}_1, w_3)$ -border. The arrow indicates the region that is eventually w_3 -periodic.



(B) $\alpha|_{G_\omega^{(a-1)}}$ is vertically periodic. Translations of \mathcal{B} are shown and α is eventually periodic on each translated set.



(C) The semi-infinite $(\tilde{\mathcal{S}}_1, w_3)$ -strip is eventually w_3 -periodic and so any η -coloring of its w_3 -extension is also w_3 -periodic (possibly of larger period), by Proposition 4.14. The arrows indicate the direction of eventual periodicity and the possibly different periods.



(D) $\alpha|_{G_\omega^{(a-1)}}$ is vertically periodic and so there is an infinite convex region where α is doubly periodic. K is the largest convex set for which this holds.

FIGURE 7. Construction of K

region where α is singly (but not doubly) periodic (Section 5.3.1) and then producing a generating set with particular properties that imply the bounds (Sections 5.3.2 and 5.3.3). The existence of this type of generating set strongly relies on the bound of $P_\eta(n, k) \leq \frac{nk}{2}$.

5.3.1. *Periodic extensions.* We show that the region K on which α is doubly periodic can be extended to a larger region on which α is singly, but not doubly, periodic.

Let $K_0 := K$. Since K has a semi-infinite edge parallel to w_3 , $\text{Ext}_{w_3}(K) \neq K$ and there exist semi-infinite lines $\ell_1, \dots, \ell_{f_1}$ parallel to w_3 such that

$$\text{Ext}_{w_3}(K) \setminus K = \bigsqcup_{i=1}^{f_1} (\mathbb{Z}^2 \cap \ell_i).$$

For $1 \leq i \leq f_1$, set $K_i := K \cup \ell_1 \cup \dots \cup \ell_i$. We continue inductively: having defined integers f_1, \dots, f_j and sets $K_1, \dots, K_{f_1+\dots+f_j}$ such that $K_{f_1+\dots+f_j}$ contains a semi-infinite boundary edge parallel to w_3 , there exist an integer f_{j+1} and semi-infinite lines $\ell_{f_1+\dots+f_j+1}, \dots, \ell_{f_1+\dots+f_{j+1}}$ such that

$$\text{Ext}_{w_3}(K_{f_1+\dots+f_j}) \setminus K_{f_1+\dots+f_j} = \bigsqcup_{i=1}^{f_{j+1}} (\mathbb{Z}^2 \cap \ell_{f_1+\dots+f_j+i})$$

By the second claim of Proposition 4.14, the restriction of α to $\bigcup_{i=1}^{\infty} K_i$ is w_3 -periodic with period at most $2|w_3 \cap \mathcal{S}| - 2$.

5.3.2. *A thin generating set.* We use the assumption on complexity $P_\eta(n, k) \leq \frac{nk}{2}$ to show that we have a generating set with a small diameter (recall Definition 3.9).

Let x_{\min} and x_{\max} denote the minimal and maximal x -coordinates of elements of \mathcal{S} . Let $d := \lfloor \frac{x_{\max} - x_{\min} + 1}{2} \rfloor$ and let the left subset of \mathcal{S} be defined by

$$\mathcal{S}_L := \{(x, y) \in \mathcal{S} : x_{\min} \leq x \leq d\}.$$

If $|\mathcal{S}_L| \geq \frac{1}{2}|\mathcal{S}|$, then by Lemma 2.6, $D_\eta(\mathcal{S}_L) \leq 0$. Let $\mathcal{S}_2 \subset \mathcal{S}_L$ be an η -generating set. Otherwise $D_\eta(\mathcal{S} \setminus \mathcal{S}_L) \leq 0$. In this case, let $\mathcal{S}_2 \subset (\mathcal{S} \setminus \mathcal{S}_L)$ be an η -generating set. In both cases, let $u \in E(\mathcal{S}_2)$ be the edge parallel to w_3 (which exists by rigidity of w_3 and Lemma 3.3) and let $v \in E(\mathcal{S}_2)$ be the edge parallel to w_1 . By construction

$$(6) \quad \text{diam}_v(\mathcal{S}_2) \leq \left\lceil \frac{\text{diam}_v(\mathcal{S})}{2} \right\rceil.$$

We call the set \mathcal{S}_2 a *thin generating set* for η .

5.3.3. *Bounding the periods of $\alpha|_K$.* Using the generating set \mathcal{S}_2 and the construction of α , we obtain strong bounds on periods of $\alpha|_K$.

Notation 5.1. There are two distinguished semi-infinite strips in K and we label them:

- Let \mathcal{T}_1 denote the $(\mathcal{S} \setminus \{w_3\}, w_3)$ -border of K .
- Let \mathcal{T}_2 denote the $(\mathcal{S} \setminus \{w_1\}, w_1)$ -border of K .

In the remainder of this section, we show:

Claim 5.2. *Maintaining notation as above,*

- (i) *The w_1 -period of $\alpha|_K$ is at most $\lfloor \frac{|w_1 \cap \mathcal{S}|}{2} \rfloor$;*
- (ii) *The w_3 -period of $\alpha|_K$ is at most $|w_3 \cap \mathcal{S}| - 1$.*

By convexity, \mathcal{S} is either w_3 - or w_4 -balanced. We prove the claim by considering three cases separately.

Case 1: Suppose \mathcal{S} is w_3 -balanced. It is immediate that $|w_3 \cap \mathbb{Z}^2| \leq |w_4 \cap \mathbb{Z}^2|$. By (2), \mathcal{S} is also w_1 -balanced. In this case we show the claimed bound on the w_1 -period of $\alpha|_K$ but prove (the stronger bound) that the w_3 -period of $\alpha|_K$ is at most $\left\lfloor \frac{|w_3 \cap \mathcal{S}|}{2} \right\rfloor$.

By maximality of K , $\alpha|_K$ is (\mathcal{S}, w_3, η) -ambiguous. Since $\alpha|_K$ is doubly periodic, $\alpha|_{\mathcal{T}_1}$ is periodic. Thus by Corollary 4.15, $\alpha|_{\mathcal{T}_1}$ is periodic with period vector parallel to w_3 and period at most $\left\lfloor \frac{|w_3 \cap \mathcal{S}|}{2} \right\rfloor$. By vertical periodicity of $\alpha|_K$, there is some $p \in \mathbb{N}$ such that the colorings $(T^{(0,mp)}\alpha)|_{\mathcal{T}_1}$ coincide for all $m = 0, 1, 2, \dots$

Again by maximality of K , $\alpha|_K$ is (\mathcal{S}, w_1, η) -ambiguous and so $\alpha|_{\mathcal{T}_2}$ is vertically periodic with period at most $\left\lfloor \frac{|w_1 \cap \mathcal{S}|}{2} \right\rfloor$. Then there is some $q \in \mathbb{N}$ such that the colorings $(T^{-mq \cdot w_3}\alpha)|_{\mathcal{T}_1}$ coincide for all $m = 0, 1, 2, \dots$ (here w_3 is understood as a vector rather than a line segment).

Since \mathcal{T}_1 is a semi-infinite $(\mathcal{S} \setminus \{w_3\}, w_3)$ -strip and \mathcal{T}_2 is a semi-infinite $(\mathcal{S} \setminus \{w_1\}, w_1)$ -strip, there exist $m_1, m_2 \in \mathbb{N}$ such that

$$P := (\mathcal{T}_1 - (0, m_1 p)) \cap (\mathcal{T}_2 - m_2 q \cdot w_3) \cap \mathbb{Z}^2$$

is the intersection of \mathbb{Z}^2 with a parallelogram, with sides parallel to w_1 and w_3 and integer vertices. This is illustrated in Figure 8.

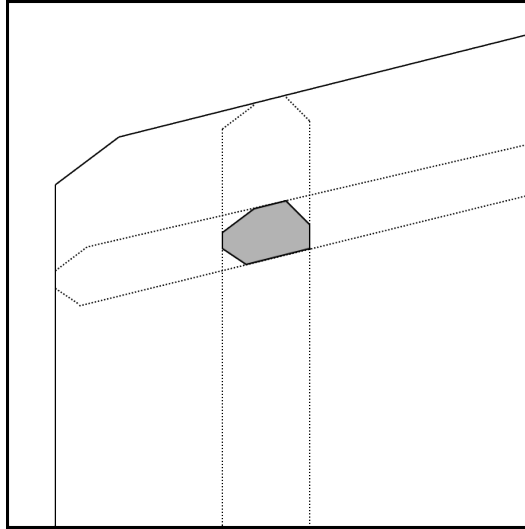


FIGURE 8. The set \mathcal{S} is the shaded convex set and K is the largest convex region shown. The set $\mathcal{T}_1 - (0, m_1 p)$ is the “diagonal” strip and $\mathcal{T}_2 - m q \cdot w_3$ is the “vertical” strip. The restriction of α to each of the strips is periodic in the direction determined by the strip and the period is at most half of the side length of the parallelogram.

Since \mathcal{S} is w_3 -balanced, if L is any line parallel to w_3 that has nonempty intersection with P , then

$$|L \cap P| \geq |w_3 \cap \mathcal{S}| - 1.$$

Therefore, by the Fine-Wilf theorem [9] and the fact that $\alpha|(\mathcal{T}_1 - m_1 p)$ is w_3 -periodic of period at most $\left\lfloor \frac{|w_3 \cap \mathcal{S}|}{2} \right\rfloor$ (by ambiguity), if L is any line parallel to w_3 that has nonempty intersection with P then there is a unique \mathcal{A} -coloring of $L \cap \mathbb{Z}^2$ that coincides with α on $L \cap P$ and is periodic of period at most $\left\lfloor \frac{|w_3 \cap \mathcal{S}|}{2} \right\rfloor$. Since \mathcal{S} is also w_1 -balanced, a similar result holds for lines parallel to w_1 : if L is any line parallel to w_1 that has nonempty intersection with P then

$$|L \cap P| \geq |w_1 \cap \mathcal{S}| - 1.$$

Therefore, since $\alpha|(\mathcal{T}_2 - m_2 q \cdot w_3)$ is w_1 -periodic of period at most $\left\lfloor \frac{|w_1 \cap \mathcal{S}|}{2} \right\rfloor$, for such an L there is at most one \mathcal{A} -coloring of $L \cap \mathbb{Z}^2$ that coincides with α on $L \cap P$ and has period at most $\left\lfloor \frac{|w_1 \cap \mathcal{S}|}{2} \right\rfloor$. Let \mathcal{C}_1 be the union of all lines parallel to w_3 that have nonempty intersection with P and let \mathcal{C}_2 be the union of all lines parallel to w_1 that have nonempty intersection with P . Let $\beta : \mathcal{C}_1 \cup \mathcal{C}_2 \rightarrow \mathcal{A}$ be the coloring just described. We claim that β extends uniquely to an \mathcal{A} -coloring of \mathbb{Z}^2 that is w_1 -periodic of period at most $\left\lfloor \frac{|w_1 \cap \mathcal{S}|}{2} \right\rfloor$ and w_3 -periodic of period at most $\left\lfloor \frac{|w_3 \cap \mathcal{S}|}{2} \right\rfloor$. Indeed, if L is any line parallel to w_1 that has nonempty intersection with \mathbb{Z}^2 , then $|L \cap (\mathcal{C}_1 \cup \mathcal{C}_2)| \geq |w_1 \cap \mathcal{S}| - 1$ (since this is true for P and \mathcal{C}_1 is produced by translating P along the vector w_3). The coloring $\beta|_{L \cap \mathcal{C}_1}$ is w_1 -periodic (in the sense of Fine and Wilf) of period at most $\left\lfloor \frac{|w_1 \cap \mathcal{S}|}{2} \right\rfloor$. As above, the coloring extends uniquely. We set $\beta : \mathbb{Z}^2 \rightarrow \mathcal{A}$ to be the coloring obtained by this procedure, and the w_3 -bound follows from w_3 -periodicity of $\beta|_{\mathcal{T}_1 - (0, m_1 p)}$ and vertical periodicity.

We claim that $\alpha|_K = \beta|_K$, which establishes the claim for the first case.

Notation 5.3. Let \mathcal{T}_3 denote the $(\mathcal{S}_2 \setminus \{v\}, v)$ -border of $(\mathcal{T}_2 + m_2 q \cdot w_3)$, where \mathcal{S}_2 is the thin generating set of Section 5.3.2 and $v \in E(\mathcal{S})$ is the edge parallel to w_1 .

The uniqueness involved in the construction of β and the bounds established above on the periods of $\alpha|(\mathcal{T}_1 - (0, m_1 p))$ and $\alpha|(\mathcal{T}_2 - m_2 q \cdot w_3)$ imply that the restrictions $\alpha|_{\mathcal{C}_1 \cup \mathcal{C}_2} = \beta|_{\mathcal{C}_1 \cup \mathcal{C}_2}$. Let $\vec{s} \in \mathbb{Z}^2$ be the shortest w_3 -period of β . Then $(T^{\vec{s}}\beta)|_{\mathcal{T}_3} = \beta|_{\mathcal{T}_3}$. Since $\text{diam}_v(\mathcal{S}_2) \leq \left\lceil \frac{\text{diam}_v(\mathcal{S})}{2} \right\rceil$ by (6), $\text{diam}_v(\mathcal{T}_3) \leq \left\lfloor \frac{\text{diam}_v(\mathcal{S})}{2} \right\rfloor$. Therefore $(\mathcal{T}_3 + \vec{s}) \subseteq (\mathcal{T}_2 + m_2 q \cdot w_3)$, and it follows that $(T^{\vec{s}}\alpha)|_{\mathcal{T}_3} = \alpha|_{\mathcal{T}_3}$. Recall that by construction, $\mathcal{S}_3 \subseteq \mathcal{S}_L \subseteq \mathcal{S}$. Since we know that $\alpha|_{\mathcal{C}_2} = \beta|_{\mathcal{C}_2}$; $\text{diam}_{w_3}(\mathcal{S}_2) \leq \text{diam}_{w_3}(\mathcal{C}_2)$; $\alpha|(\mathcal{T}_2 - m_2 q \cdot w_3)$ is w_3 -periodic (in the sense of Fine and Wilf); $\beta|_{\mathcal{C}_2}$ is w_3 -periodic with period \vec{s} ; and \mathcal{S}_2 is an η -generating set, we can conclude that the coloring $\alpha|_K$ can be determined from $\alpha|((\mathcal{T}_1 - (0, m_1 p)) \cap (\mathcal{T}_2 - m_2 q \cdot w_3))$ and it follows by induction that $\alpha|_K = \beta|_K$.

Case 2: Suppose \mathcal{S} is w_4 -balanced and for infinitely many m_1 , $\alpha|_{\mathcal{T}_1 - (0, m_1 p)}$ extends non-uniquely to its w_4 -extension. For any such m_1 , by Corollary 4.15 $\alpha|_{\mathcal{T}_1 - (0, m_1 p)}$ is periodic with period vector parallel to w_4 and period at most $\left\lfloor \frac{|w_4 \cap \mathcal{S}|}{2} \right\rfloor$. The proof now proceeds as in Case 1, with w_4 taking the role of w_3 .

Case 3: Suppose \mathcal{S} is w_4 -balanced and for all but finitely many m_1 , $\alpha|_{\mathcal{T}_1 - (0, m_1 p)}$ extends uniquely to its w_4 -extension. We have that $\alpha|_{\bigcup_{i=1}^{\infty} K_i}$ is periodic with period vector parallel to w_3 and period at most $2|w_3 \cap \mathcal{S}| - 2$, but is *not* vertically

periodic. Thus there is some semi-infinite (\mathcal{S}, w_4) -strip in $\bigcup_{i=1}^{\infty} K_i$ to which the restriction of α is w_4 -ambiguous, as otherwise each of the finitely many η -colorings arising as the restriction of α to such strips extend uniquely to their w_4 -extension, forcing vertical periodicity. Let \mathcal{T}_4 be a semi-infinite $(\mathcal{S} \setminus \{w_4\})$ -strip in $\bigcup_{i=1}^{\infty} K_i$ to which the restriction of α is w_4 -ambiguous. Without loss of generality, we can assume that for any $p > 0$, $\alpha|_{\mathcal{T}_4 - (0, p)}$ extends uniquely to its w_4 -extension. By Corollary 4.15, $\alpha|_{\mathcal{T}_4}$ is eventually periodic with period vector parallel to w_4 and period at most $\left\lfloor \frac{|w_4 \cap \mathcal{S}|}{2} \right\rfloor$ and gap at most $|w_4 \cap \mathcal{S}| - 1$. Again by Corollary 4.15, the restriction of α to the w_4 -extension of \mathcal{T}_4 is eventually periodic with the same gap and period at most $2 \left\lfloor \frac{|w_4 \cap \mathcal{S}|}{2} \right\rfloor \leq |w_4 \cap \mathcal{S}| \leq |w_3 \cap \mathcal{S}| - 1$. Inductively, we produce a sequence of sets

$$\mathcal{T}_4 = \mathcal{T}_4^1 \subset \mathcal{T}_4^2 \subset \dots$$

where \mathcal{T}_4^{i+1} is the w_4 -extension of \mathcal{T}_4^i . Since $\alpha|_{\mathcal{T}_4^i}$ extends uniquely to its w_4 -extension and since $\alpha|_{\mathcal{T}_4}$ is periodic with period at most $|w_3 \cap \mathcal{S}| - 1$, the restriction of \mathcal{T}_4^i is also w_3 -periodic with period dividing that of $\alpha|_{\mathcal{T}_4}$, for all $i = 1, 2, \dots$. Since $\alpha|_K$ is doubly periodic and

$$K \cap \bigcup_{i=1}^{\infty} \mathcal{T}_4^i$$

is an infinite, convex set whose two semi-infinite edges are non-parallel, the restriction of α to any $(\mathcal{S} \setminus \{w_4\}, w_4)$ -strip is periodic with period vector parallel to w_4 and period dividing the period of $\alpha|_{\mathcal{T}_4}$. Since $\alpha|_{\mathcal{T}_2 + m_2 \cdot qw_3}$ is vertically periodic with period at most $\left\lfloor \frac{|w_1 \cap \mathcal{S}|}{2} \right\rfloor$, has w_3 -diameter at least $\text{diam}_{w_3}(P)$, and the w_3 -period of $\alpha|_K$ is at most $|w_3 \cap \mathcal{S}| - 1$, $\alpha|_K$ is also vertically periodic of period at most $\left\lfloor \frac{|w_1 \cap \mathcal{S}|}{2} \right\rfloor$.

This completes the proof of Claim 5.2.

5.4. Completing the proof of Theorem 1.5. We make use of the properties of α to obtain a contradiction. Specifically, we show that for a given η -generating set \mathcal{S} , there exists a convex subset $\mathcal{S}^* \subset \mathcal{S}$ for which there are more than

$$P_{\eta}(\mathcal{S}) - P_{\eta}(\mathcal{S}^*)$$

η -colorings of \mathcal{S}^* that extend non-uniquely to η -colorings of \mathcal{S} . This leads to a contradiction, as if

$$\begin{aligned} P &:= \{(T^{\vec{u}}\eta)|_{\mathcal{S}} : \vec{u} \in \mathbb{Z}^2\}; \\ Q &:= \{(T^{\vec{u}}\eta)|_{\mathcal{S}^*} : \vec{u} \in \mathbb{Z}^2\}, \end{aligned}$$

then there is a natural surjective map $R : P \rightarrow Q$ by restriction. The number of elements of Q that have more than one preimage (equivalently, the number of colorings of \mathcal{S}^* that extend nonuniquely to colorings of \mathcal{S}) is at most $|P| - |Q| = P_{\eta}(\mathcal{S}) - P_{\eta}(\mathcal{S}^*)$.

5.4.1. Construction of the set \mathcal{S}^* . Given $x \in \mathbb{Z}$, let $\ell_x = \{(x, y) \in \mathbb{Z}^2 : y \in \mathbb{Z}\}$ denote the vertical line passing through x . For $x \in \mathbb{Z}$ such that $\ell_x \cap \mathcal{S} \neq \emptyset$, let A_x denote the bottom-most $|w_1 \cap \mathcal{S}| - 2$ elements of $\ell_x \cap \mathcal{S}$ (recall that \mathcal{S} is w_1 -balanced

and so each such intersection contains at least $|w_1 \cap \mathcal{S}| - 1$ integer points). Given $d \geq 1$, define

$$(7) \quad B(d) := \bigcup_{i=0}^{d-1} A_{(x_{\max}-i)},$$

where, as in Section 5.3.2, x_{\max} denotes the maximal x -coordinate of any element of \mathcal{S} . Let $\mathcal{T}(K) := \{\vec{u} \in \mathbb{Z}^2 : \mathcal{S} + \vec{u} \subset K\}$ be the set of translations taking \mathcal{S} to a subset of K . Choose minimal $d \geq 1$ such that for any $\vec{u}, \vec{v} \in \mathcal{T}(K)$, whenever $\alpha|_{B(d)} + \vec{u} = \alpha|_{B(d)} + \vec{v}$, we have that $\alpha|_{\mathcal{S} + \vec{u}} = \alpha|_{\mathcal{S} + \vec{v}}$. Since $\alpha|_K = \beta|_K$ and β is doubly periodic, we can rephrase this condition as saying that d is minimal such that

$$(8) \quad \text{every } \beta\text{-coloring of } B(d) \text{ extends uniquely to a } \beta\text{-coloring of } \mathcal{S}.$$

(Note that such an integer d exists because $\alpha|_K$ is vertically periodic with period at most $\left\lfloor \frac{|w_1 \cap \mathcal{S}|}{2} \right\rfloor \leq |w_1 \cap \mathcal{S}| - 2$.) Let $\mathcal{S}^* \subset \mathcal{S}$ be the set obtained by removing the topmost element of $\ell_x \cap \mathcal{S}$ for all x . Note that \mathcal{S}^* is a convex, proper subset of \mathcal{S} . Therefore $B(d) \subseteq \mathcal{S}^*$ and $D_\eta(\mathcal{S}^*) > D_\eta(\mathcal{S})$, by Property (iii) of Lemma 4.1. As a result, there are at most $|\mathcal{S} \setminus \mathcal{S}^*| - 1$ distinct η -colorings of \mathcal{S}^* that extend non-uniquely to η -colorings of \mathcal{S} .

In the next two sections, we show that there are at least $|\mathcal{S} \setminus \mathcal{S}^*| = \text{diam}_{w_1}(\mathcal{S})$ distinct η -colorings of \mathcal{S}^* that extend non-uniquely to η -colorings of \mathcal{S} . The colorings come from two sources: we find d such η -colorings that are of the form $(T^{\vec{x}}\beta)|_{\mathcal{S}^*}$ (Section 5.4.3) and we find $\text{diam}_{w_1}(\mathcal{S}) - d$ additional η -colorings that we show are not of the form $(T^{\vec{x}}\beta)|_{\mathcal{S}^*}$ (Section 5.4.2). We finally remark that all of these colorings are α -colorings of \mathcal{S}^* that extend non-uniquely to α -colorings of \mathcal{S} . This causes no problem since $\alpha \in \overline{\mathcal{O}(\eta)}$, so every α -coloring of \mathcal{S}^* that extends non-uniquely to an α -coloring of \mathcal{S} is also an η -coloring that extends non-uniquely.

5.4.2. Counting colorings along the w_1 -boundary. In this section, we find

$$(9) \quad \bar{d} := \text{diam}_{w_1}(\mathcal{S}) - d.$$

distinct α -colorings of \mathcal{S}^* that extend non-uniquely to α -colorings of \mathcal{S} . We show that none of these colorings are of the form $(T^{\vec{x}}\beta)|_{\mathcal{S}^*}$ for $\vec{x} \in \mathbb{Z}^2$, meaning that they are not also β -colorings of \mathcal{S}^* .

Setup. Translating the coordinate system if necessary, we can assume that the edge of $\text{conv}(K)$ parallel to w_1 is $\{(0, y) \in \mathbb{Z}^2 : y \leq 0\}$ and that the intersection of the w_1 -extension of K with the line $\{(-1, y) : y \in \mathbb{Z}\}$ is the semi-infinite line $\{(-1, y) : y \leq y_0\}$ for some $y_0 \in \mathbb{Z}$. Without loss of generality, assume that

$$w_1 = \{(-1, y) \in \mathbb{Z}^2 : y_0 - |w_1 \cap \mathcal{S}| + 1 \leq y \leq y_0\}.$$

Let \mathcal{S}^* and $B(d)$ be as in Section 5.4.1.

Let $c_1, \dots, c_t : B(d) \rightarrow \mathcal{A}$ be the set of all β -colorings of $B(d)$, and note that this set coincides with the η -colorings of $B(d)$ occurring in the set K . For $i = 1, \dots, t$, let $C_i : \mathcal{S}^* \rightarrow \mathcal{A}$ denote the unique β -coloring of \mathcal{S}^* whose restriction to $B(d)$ is c_i , and note that the uniqueness follows from (8). Equivalently, this is the coloring $(T^{\vec{u}}\alpha)|_{\mathcal{S}^*}$, where $\vec{u} \in \mathbb{Z}^2$ is chosen such that $\mathcal{S} + \vec{u} \subset K$ and $\alpha|_{B(d)} + \vec{u} = c_i$.

As in Section 5.2.1, let $\tilde{\mathcal{S}} := \mathcal{S} \setminus \{w_1\}$. Let

$$(10) \quad \vec{b} \in \mathbb{Z}^2 \text{ be the shortest } w_3\text{-period vector for } \alpha|_K.$$

Let \mathcal{T}_2 be the $(\tilde{\mathcal{S}}, w_1)$ -border of K , as in Notation 5.1. Then the colorings of \mathcal{T}_2 given by $\alpha|_{\mathcal{T}_2}$ and $(T^{\vec{b}}\alpha)|_{\mathcal{T}_2}$ coincide. By maximality of K , the colorings of $\mathcal{T}_2 \cup \{(-1, y) : y \leq y_0\}$ given by α and $T^{\vec{b}}\alpha$ do not coincide. We begin by comparing the colorings $\alpha|_{\{(-1, y) : y \leq y_0\}}$ and $(T^{\vec{b}}\alpha)|_{\{(-1, y) : y \leq y_0\}}$.

The line $\{(-1, y) : y \leq y_0\}$ and behavior of α . By the first part of Claim 5.2, $\alpha|_K$ is vertically periodic of period at most $\left\lfloor \frac{|w_1 \cap \mathcal{S}|}{2} \right\rfloor$.

(11) Let $(0, -p)$ be the shortest vertical period for $(T^{\vec{b}}\alpha)|_{\{(-1, y) : y \leq y_0\}}$.

Then p is a divisor of the smallest vertical period of $\alpha|_K$. In particular,

$$(12) \quad p \leq \left\lfloor \frac{|w_1 \cap \mathcal{S}|}{2} \right\rfloor.$$

Claim 5.4. $\alpha|_{\{(-1, y) : y \leq y_0\}}$ is vertically periodic with period $q \leq \left\lfloor \frac{|w_1 \cap \mathcal{S}|}{2} \right\rfloor$.

To prove the claim, we first show that there are no integers $0 \leq j_1, j_2 < p$ such that

$$(T^{(0, -j_1)}\alpha)|_{\mathcal{S}} = (T^{(0, -j_2) + \vec{b}}\alpha)|_{\mathcal{S}}.$$

For contradiction, suppose not and without loss of generality suppose $j_1 \leq j_2$. We consider the case that $j_1 < j_2$ first and then consider $j_1 = j_2$.

Suppose $j_1 < j_2$ and observe that

$$(T^{(0, -j_1)}\alpha)|_{\tilde{\mathcal{S}}} = (T^{(0, -j_2) + \vec{b}}\alpha)|_{\tilde{\mathcal{S}}}.$$

Since \vec{b} is a period vector for $\alpha|_K$,

$$(T^{(0, -j_1)}\alpha)|_{\tilde{\mathcal{S}}} = (T^{(0, -j_2)}\alpha)|_{\tilde{\mathcal{S}}}.$$

Since \mathcal{S} is w_1 -balanced, every line parallel to w_1 that has nonempty intersection with $\tilde{\mathcal{S}}$ intersects in at least $|w_1 \cap \mathcal{S}| - 1$ places. Since $\alpha|_K$ is vertically periodic of period at most $\left\lfloor \frac{|w_1 \cap \mathcal{S}|}{2} \right\rfloor \leq |w_1 \cap \mathcal{S}| - 2$, this implies that $j_2 - j_1$ is a vertical period for \mathcal{T}_2 (the $(\tilde{\mathcal{S}}, w_1)$ -border of K). By Claim 5.2, the minimal w_3 -period of $\alpha|_K$ is smaller than the w_3 -width of \mathcal{T}_2 , so $j_2 - j_1$ is a vertical period for $\alpha|_K$. This contradicts minimality of p , and we conclude that j_1 cannot be smaller than j_2 .

Suppose $j_1 = j_2$. Then since \mathcal{S} is η -generating and

$$\begin{aligned} (T^{(0, -j_1)}\alpha)|_{\mathcal{S}} &= (T^{(0, -j_1) + \vec{b}}\alpha)|_{\mathcal{S}}; \\ \alpha|_{\mathcal{T}_2} &= (T^{\vec{b}}\alpha)|_{\mathcal{T}_2}, \end{aligned}$$

we have that

$$\alpha|_{\{(-1, y) : y \leq y_0\}} = (T^{\vec{b}}\alpha)|_{\{(-1, y) : y \leq y_0\}}.$$

This contradicts maximality of K . We conclude that no such integers j_1, j_2 exist.

Now, there are at most $P_\eta(\mathcal{S}) - P_\eta(\tilde{\mathcal{S}})$ distinct η -colorings of $\tilde{\mathcal{S}}$ that extend non-uniquely to η -colorings of \mathcal{S} . Each of the colorings

$$\{(T^{(0, -j)}\alpha)|_{\tilde{\mathcal{S}}} : j \in \mathbb{N}\}$$

is such a coloring, by maximality of K and the fact that \mathcal{S} is η -generating. However,

$$(13) \quad \{(T^{(0, -j)}\alpha)|_{\mathcal{S}} : j \in \mathbb{N}\} \cap \{(T^{\vec{b} + (0, -j)}\alpha)|_{\mathcal{S}} : j \in \mathbb{N}\} = \emptyset.$$

On the other hand,

$$\begin{aligned} \left| \{(T^{(0,-j)}\alpha) \upharpoonright_{\mathcal{S}} : j \in \mathbb{N}\} \cup \{(T^{\vec{b}+(0,-j)}\alpha) \upharpoonright_{\mathcal{S}} : j \in \mathbb{N}\} \right| \\ \leq P_{\eta}(\mathcal{S}) - P_{\eta}(\tilde{\mathcal{S}}) + \left| \{(T^{(0,-j)}\alpha) \upharpoonright_{\tilde{\mathcal{S}}} : j \in \mathbb{N}\} \right|. \end{aligned}$$

Since

$$\left| \{(T^{\vec{b}+(0,-j)}\alpha) \upharpoonright_{\mathcal{S}} : j \in \mathbb{N}\} \right| \geq \left| \{(T^{(0,-j)}\alpha) \upharpoonright_{\tilde{\mathcal{S}}} : j \in \mathbb{N}\} \right|,$$

there are at most $P_{\eta}(\mathcal{S}) - P_{\eta}(\tilde{\mathcal{S}}) \leq \left\lfloor \frac{|w_1 \cap \mathcal{S}|}{2} \right\rfloor$ elements of the set

$$\{(T^{(0,-j)}\alpha) \upharpoonright_{\mathcal{S}} : j \in \mathbb{N}\}.$$

By Proposition 4.8 $\alpha \upharpoonright_{\{(-1,y) : y \leq y_0\}}$ is vertically periodic and by the above bound,

$$(14) \quad \text{the minimal vertical period of } \alpha \upharpoonright_{\{(-1,y) : y \leq y_0\}} \text{ is } q \leq \left\lfloor \frac{|w_1 \cap \mathcal{S}|}{2} \right\rfloor.$$

This establishes the claim.

Using the bounds on p and q given by (12) and (14), we establish the following claim.

Claim 5.5. *There do not exist integers $0 \leq i < \bar{d}$ and $y \leq 0$ such that*

$$(T^{(-i,-y)}\alpha) \upharpoonright_{\mathcal{S}^*} \text{ is a } \beta\text{-coloring of } \mathcal{S}^*.$$

We establish the claim by contradiction, and so choose i and y for which the claim fails. Then by definition of $\bar{d} = \text{diam}_{w_1}(\mathcal{S}) - d$, we have that $B(d) + (i, y)$ is a subset of K , and since $\alpha \upharpoonright_K = \beta \upharpoonright_K$, we have that $(T^{(-i,-y)}\alpha) \upharpoonright_{B(d)}$ is a β -coloring of $B(d)$. By definition of d , this extends uniquely to a β -coloring of \mathcal{S}^* . Thus,

$$(T^{(-i,-y)}\alpha) \upharpoonright_{\mathcal{S}^*} = (T^{(-i,-y)+\vec{b}}\alpha) \upharpoonright_{\mathcal{S}^*}.$$

Therefore,

$$(15) \quad \begin{aligned} &\text{there is a set of } |w_1 \cap \mathcal{S}| - 1 \text{ consecutive integer points} \\ &\text{on the line } \{(-1, y) : y \leq y_0\} \text{ where } \alpha \text{ and } T^{\vec{b}}\alpha \text{ coincide.} \end{aligned}$$

By (12), the vertical period of the coloring $\alpha \upharpoonright_{\{(-1, y) : y \leq y_0\}}$ is $p \leq \left\lfloor \frac{|w_1 \cap \mathcal{S}|}{2} \right\rfloor$ and by (14) the vertical period of the coloring $(T^{\vec{b}}\alpha) \upharpoonright_{\{(-1, y) : y \leq y_0\}}$ is $q \leq \left\lfloor \frac{|w_1 \cap \mathcal{S}|}{2} \right\rfloor$.

If $p = q$, then $\alpha \upharpoonright_{\{(-1, y) : y \leq y_0\}} = (T^{\vec{b}}\alpha) \upharpoonright_{\{(-1, y) : y \leq y_0\}}$, contradicting maximality of K . Otherwise $p \neq q$, and since both are integers, $p + q - \gcd(p, q) \leq |w_1 \cap \mathcal{S}| - 2$. By the Fine-Wilf Theorem [9] and (15), we again have that

$$\alpha \upharpoonright_{\{(-1, y) : y \leq y_0\}} = (T^{\vec{b}}\alpha) \upharpoonright_{\{(-1, y) : y \leq y_0\}},$$

again a contradiction. We conclude that no such $0 \leq i < \bar{d}$ and $y \leq 0$ exist, establishing the claim.

If there were some $j = 0, \dots, \bar{d} - 1$ such that the restriction of α to the strip given by

$$\bigcup_{s \in \mathbb{Z}} (\tilde{\mathcal{S}} + (-j, s))$$

is vertically periodic, then η would be vertically periodic by Corollary 4.9, a contradiction. On the other hand, by Corollary 4.9, the restriction of α to each such strip

is eventually vertically periodic, since $\alpha|_K$ is. Therefore for all $j = 0, \dots, \bar{d} - 1$, there exists maximal $s_j \in \mathbb{Z}$ such that

$$(16) \quad \text{the restriction of } \alpha \text{ to } \bigcup_{s=-\infty}^{s_j-1} (\mathcal{S} + (-j, s)) \text{ is vertically periodic.}$$

Counting α -colorings of \mathcal{S}^* that extend non-uniquely.

Claim 5.6. *With the integers $\{s_j\}_{j=0}^{\bar{d}-1}$ as defined above,*

- (i) *the η -colorings of \mathcal{S}^* given by $\alpha|(\mathcal{S}^* + (-j, s_j))$ are distinct for $j = 1, \dots, \bar{d}$;*
- (ii) *for each such j , the coloring of \mathcal{S}^* given by $\alpha|(\mathcal{S}^* + (-j, s_j))$ extends non-uniquely to an α -coloring of \mathcal{S} .*

We begin by establishing the first statement. Observe that $(B(d) - (j, 0)) \subset K$. By maximality of s_j , the coloring $(T^{(j, -s_j)}\alpha)|_{B(d)}$ is a β -coloring of $B(d)$. Therefore there is some $i = 1, \dots, t$ such that $(T^{(j, -s_j)}\alpha)|_{B(d)} = c_i$ and C_i is the unique β -coloring of \mathcal{S}^* whose restriction to $B(d)$ is c_i . We claim that $(T^{(j, -s_j)}\alpha)|_{\mathcal{S}^*} \neq C_i$.

For each $j \geq 0$, the coloring $(T^{(0, -j)+\bar{b}}\alpha)|_{\mathcal{S}}$ is a β -coloring of \mathcal{S} since $\alpha|_K = \beta|_K$. By (13), none of the colorings $\{(T^{(0, -j)}\alpha)|_{\mathcal{S}} : j \in \mathbb{N}\}$ are β -colorings. By (14) and maximality of s_j , $\alpha|_{\{(-1, s_j - y) : y \in \mathbb{N} \cup \{0\}\}}$ is vertically periodic with period at most $\left\lfloor \frac{|w_1 \cap \mathcal{S}|}{2} \right\rfloor$. Since every vertical line that has nonempty intersection with \mathcal{S}^* intersects in at least $|w_1 \cap \mathcal{S}| - 2$ integer points, the restrictions of α and β to the set $\{(-1, y) : y \in \mathbb{Z}^2\} \cap (\mathcal{S}^* + (-j, s_j))$ cannot coincide (otherwise by the Fine-Wilf Theorem they would coincide everywhere on the semi-infinite line). On the other hand, the restrictions of α and β to $\{(x, y) \in \mathbb{Z}^2 : x \geq 0\} \cap (\mathcal{S}^* + (-j, s_j))$ do coincide, since they agree on K and s_j was chosen such that the semi-infinite \mathcal{S} -strip below it was vertically periodic. Consequently, the rightmost vertical line where $\alpha|_{\mathcal{S}^* + (-j, s_j)}$ differs from $\beta|_{\mathcal{S}^* + (-j, s_j)}$ has x -coordinate $x_{\min} + j - 1$. Therefore, for distinct $1 \leq j_1 < j_2 \leq \bar{d}$, the η -colorings of \mathcal{S}^* given by $\alpha|(\mathcal{S}^* + (-j_1, s_{j_1}))$ and $\alpha|(\mathcal{S}^* + (-j_2, s_{j_2}))$ are distinct.

For the second statement, by (16) the restriction of α to the semi-infinite \mathcal{S} -strip given by

$$\bigcup_{s=-\infty}^{s_j-1} (\mathcal{S} + (-j, s))$$

is vertically periodic and s_j is the largest integer with this property. If the vertical period is $p \in \mathbb{N}$, then by periodicity, the η -colorings of \mathcal{S}^* given by $\alpha|_{\mathcal{S}^* + (-j, s_j)}$ and $\alpha|_{\mathcal{S}^* + (-j, s_j - p)}$ coincide. But by maximality of s_j , the η -colorings of \mathcal{S}^* given by the functions $\alpha|_{\mathcal{S} + (-j, s_j)}$ and $\alpha|_{\mathcal{S} + (-j, s_j - p)}$ are distinct, establishing the claim.

In total, we have counted $\text{diam}_{w_1}(\mathcal{S}^*) - d$ distinct η -colorings of \mathcal{S}^* that extend non-uniquely to η -colorings of \mathcal{S} . Moreover, for each such coloring, the coloring of \mathcal{S}^* was

$$(17) \quad \text{not of the form } (T^{\vec{x}}\beta)|_{\mathcal{S}^*} \text{ for any } \vec{x} \in \mathbb{Z}^2,$$

since there was a vertical line in \mathcal{S}^* where the coloring can be distinguished from the β -coloring induced from its restriction to $B(d)$.

5.4.3. Counting colorings along the w_3 -boundary. In this section we find d distinct α -colorings of \mathcal{S}^* that extend non-uniquely to α -colorings of \mathcal{S} . Each of these colorings is of the form $(T^{\vec{x}}\beta)|_{\mathcal{S}^*}$ for some $\vec{x} \in \mathbb{Z}^2$, and hence they are all distinct from those found in Section 5.4.2.

Recall that \mathcal{T}_1 , as defined in Notation 5.1 is the $(\mathcal{S} \setminus \{w_3\}, w_3)$ -border of K and that the restriction $\alpha|_{\mathcal{T}_1}$ is periodic with period vector parallel to w_3 . Fix $\vec{d} \in \mathbb{Z}^2$ such that the sets $\{(\mathcal{S} \setminus \{w_3\}) + \vec{d} + iw_3 : i = -1, 0, 1\}$ are subsets of \mathcal{T}_1 , but none of the sets $\{\mathcal{S} + \vec{d} + iw_3 : i = -1, 0, 1\}$ are.

Let $A, B \in \mathbb{Z}$ denote the minimal and maximal x -coordinates of elements of w_3 , respectively. Enumerate the elements of $\mathcal{S} \setminus \mathcal{S}^*$ whose x -coordinates are between A and B as $z_1, \dots, z_{\text{diam}_{w_1}(\mathcal{S} - A + 1)}$, where the x -coordinate of z_{i+1} is always larger than that of z_i . Since β is w_3 -periodic with period at most $|w_3 \cap \mathcal{S}| - 1$ (by Claim 5.2) it follows that $d \leq B - A + 1$. Let $\vec{u}_1, \dots, \vec{u}_d \in \mathbb{Z}^2$ be the vectors $\vec{u}_i = z_1 - z_i$. Observe that $(\mathcal{S}^* + \vec{d} + \vec{u}_i) \subset \mathcal{T}_1 \subset K$ for $i = 1, \dots, d$. For $i = 1, \dots, d$, we claim that the η -colorings of \mathcal{S}^* given by $\alpha|_{\mathcal{S}^* + \vec{d} + \vec{u}_i}$ are distinct. If not, suppose that the colorings given by $\alpha|_{\mathcal{S}^* + \vec{d} + \vec{u}_{\text{diam}_{w_1}(\mathcal{S}^*) - j_1}}$ and $\alpha|_{\mathcal{S}^* + \vec{d} + \vec{u}_{\text{diam}_{w_1}(\mathcal{S}^*) - j_2}}$ coincide for some $1 \leq j_1 < j_2 \leq d$. Then $0 < j_2 - j_1 < d - 1$. By (7), $B(d)$ is the intersection of \mathbb{Z}^2 with the disjoint union of vertical line segments, each of which contains at least $|w_1 \cap \mathcal{S}| - 2$ integer points. Since the vertical period of β is at most $|w_1 \cap \mathcal{S}| - 2$, we have that the vector $u_{j_2} - u_{j_1}$ must be a period vector for β , and the x -component of this vector is $j_2 - j_1 \leq d - 1$. Thus any β -coloring of \mathcal{S} can be deduced from the β -coloring of $B(j_2 - j_1)$, contradicting the minimality of d .

Finally, since $\alpha|_K$ is vertically periodic, $(\mathcal{S}^* + \vec{d} + u_i) \subset K$, and $(\mathcal{S} + \vec{d} + u_i) \not\subset K$, the η -colorings of \mathcal{S}^* given by $\alpha|_{(\mathcal{S}^* + \vec{d} + u_i)}$ and $\alpha|_{(\mathcal{S}^* + \vec{d} + u_{i-(0,P)})}$ coincide (where P is the minimal vertical period of $\alpha|_K$). The η -colorings of \mathcal{S} given by $\alpha|_{(\mathcal{S} + \vec{d} + u_i)}$ and $\alpha|_{(\mathcal{S} + \vec{d} + u_{i-(0,p)})}$ cannot coincide, by maximality of K and Corollary 3.7. Therefore we obtain at least d distinct η -colorings of \mathcal{S}^* that extend non-uniquely to η -colorings of \mathcal{S} that are of the form $(T^{\vec{u}}\beta)|_{\mathcal{S}^*}$ for some $\vec{u} \in \mathbb{Z}^2$.

5.4.4. Total number of colorings. In Sections 5.4.2 and 5.4.3, we have described at least $\text{diam}_{w_1}(\mathcal{S}^*)$ distinct η -colorings of \mathcal{S}^* that extend non-uniquely to η -colorings of \mathcal{S} . However, this produces more than $P_\eta(\mathcal{S}) - P_\eta(\mathcal{S}^*) \leq \text{diam}_{w_1}(\mathcal{S}^*) - 1$ (since the discrepancy of \mathcal{S}^* is larger than that of \mathcal{S}) colorings of \mathcal{S}^* that extend non-uniquely to colorings of \mathcal{S} , the desired contradiction. This completes the proof of Theorem 1.5. \square

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